# Problem Set 10

November 29, 2025.

**1.** Suppose $f$ is bounded and analytic in $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, extends continuously to the real axis, and is real-valued on the real axis. Prove that $f$ is constant.

**2.** Let $f$ be an entire function, which is real on the real axis and imaginary on the imaginary axis. Prove that $f$ is an odd function (i.e., $f(-z) = -f(z)$ for all $z \in \mathbb{C}$).

**3.** Define a function $f$ analytic in $\mathbb{C} \setminus \{x + iy : x \leq 0, y = 0\}$ and such that $f(x) = x^x$ for all real $x > 0$. Find $f(i)$ and $f(-i)$. Show that $f(\bar{z}) = \overline{f(z)}$, for all $z$ in the domain of $f$.

**4.** (a) Prove that the function $f(z) = \dfrac{z-i}{z+i}$ is an analytic bijection between the upper half-plane $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and the open unit disc, with an analytic inverse $g(w) = -i \cdot \dfrac{w+1}{w-1}$.

(b) Use part (a) and the Schwarz Reflection Principle to prove that there is no non-constant analytic function in the open unit disc (with a continuous extension to the unit circle), which is real-valued on the unit circle.

(c) Prove that there is no function $f$ analytic in the disc $\{z \in \mathbb{C} : |z| < 2025\}$ and such that $|f(z)| \longrightarrow \infty$ as $|z| \longrightarrow 2025^-$.

**5.** Suppose $f$ is analytic in the upper open unit semi-disc $\Omega = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im}(z) > 0\}$ and extends continuously to $\overline{\Omega}$. Explain why it is not possible that $f(x) = |x|$ for all real values of $x$ (with $|x| \leq 1$).

**6.** Let $f$ be a function analytic in the open unit disc $D(0;1)$, and assume that $|f(z)| < 1$ for all $z \in D(0;1)$. Prove that, if there exist two distict points $a, b \in D(0;1)$ with $f(a) = a$ and $f(b) = b$, then $f(z) = z$ for all $z \in D(0;1)$.