Problem Set 1
September 6, 2025.

1. Express in the form $x + iy$ (with $x, y \in \mathbb{R}$):

(a) $\frac{1}{5+3i}$

(b) $\frac{(2+i)(3+2i)}{i-1}$

(c) $\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^4$.

2. Let $P \in \mathbb{R}[z]$ be a polynomial with real coefficients. Show that $\overline{P(z)} = P(\bar{z})$.

3. Let $P \in \mathbb{R}[z]$ be a polynomial with real coefficients. Prove that $P(z_0) = 0$ if and only if $P(\bar{z}_0) = 0$; i.e., show that the roots of P come in conjugate pairs.

4. Let ζ be an nth root of unity and $\zeta \neq 1$. Show that ζ satisfies the equation $z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$.

5. Describe the sets in the plane whose points satisfy the following relations:

(a) $|z - i| \leq 2$

(b) $\left|\frac{z-1}{z+1}\right| = 1$

(c) $|z - 2| > |z - 3|$

(d) $|z| > 2$ and $\mathrm{Im}(z) < 0$.

6. Let z be a point on the unit circle, with $\mathrm{Im}(z) \neq 0$. Show that $\mathrm{Arg}\left(\frac{z-1}{z+1}\right)$ is either $\pi/2$ or $-\pi/2$ depending on whether $\mathrm{Im}(z) > 0$ or $\mathrm{Im}(z) < 0$.

7. Assuming $|z| = 1$ and $\bar{z}w \neq 1$, show that

$$\left|\frac{z - w}{1 - \bar{z}w}\right| = 1\,.$$

8. Suppose that pairwise distinct $z_1, z_2, z_3 \in \mathbb{C}$ satisfy the equality

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{z_1 - z_3}{z_2 - z_3}\,.$$

Prove that then z_1, z_2 and z_3 are the vertices of an equilateral triangle, that is,

$$|z_1 - z_2| = |z_1 - z_3| = |z_2 - z_3|\,.$$