Problem Set 2

September 13, 2025.

1. Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$, for all $z_1, z_2 \in \mathbb{C}$, and interpret the result geometrically.

2. Recall that, for $z = |z|(\cos\vartheta + i\sin\vartheta)$ and $w = |w|(\cos\varphi + i\sin\varphi)$, one has

$$zw = |zw|(\cos(\vartheta+\varphi) + i\sin(\vartheta+\varphi))\,.$$

Use induction to prove that $z^n = |z|^n(\cos(n\vartheta) + i\sin(n\vartheta))$, for $z \in \mathbb{C}$ as above and all $n \in \mathbb{Z}_+$.

3. Solve the following equations in polar form and locate the roots in the complex plane:

(a) $z^6 = 1$

(b) $z^4 = -1$

(c) $z^5 = -1 + i\sqrt{3}$.

4. Let $P(z) = a_0 z^n + \cdots + a_{n-1}z + a_n$ be a polynomial with complex coefficients, where $a_0 \neq 0$, $n \in \mathbb{Z}_+$, and let $R = \max\{\sqrt[k]{\frac{|a_k|}{|a_0|}} \,:\, k = 1, \ldots, n\}$. Prove that, if $P(z_0) = 0$ then $|z_0| \leq 2R$.

5. Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, where $a_0, \ldots, a_n \in \mathbb{R}$, $n \in \mathbb{Z}_+$, $a_n \neq 0$, and $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$. Show that all the roots of $P(z)$ lie in the closed unit disc (that is, if $P(z_0) = 0$ then $|z_0| \leq 1$). [Hint: Consider the polynomial $(1-z)P(z)$.]

6. For $n \geq 3$, let D_n be a regular n-gon inscribed in the unit circle. Suppose that one of the vertices of D_n is at 1, and let $d_1, d_2, \ldots, d_{n-1}$ be the distances of the other vertices from 1. Prove that $d_1 d_2 \ldots d_{n-1} = n$.

7. Let $P(z)$ be a non-constant polynomial. Prove that $\lim_{z\to\infty} P(z) = \infty$ (that is, prove that

$$\forall R > 0\ \exists r > 0,\ |z| > r \Longrightarrow |P(z)| > R\,.)$$

8. Prove that every open connected set S in $\mathbb{C}$ is path-connected. [Hint: Given $z_0 \in S$, consider the set S_0 of points in S which can be connected by a path with z_0. Show that S_0 is non-empty, open, and closed in S. Use definition of connectedness to derive that $S_0 = S$.]