Problem Set 3
September 20, 2025.

1. Determine convergence of the series $\sum_{n=1}^{\infty} \frac{1+i^{2n}}{n^n}$.

2. Let $S \subset \mathbb{C}$ be a non-empty set, and let $f_n : S \to \mathbb{C}$, $n \in \mathbb{N}$, be functions on S. Suppose that all the f_n are continuous, and the sequence $(f_n)_n$ converges locally uniformly to a function $f : S \to \mathbb{C}$. Prove that f is continuous.

3. Let $S \subset \mathbb{C}$ be a non-empty set, and let $f_n : S \to \mathbb{C}$ ($n \in \mathbb{N}$) and $f : S \to \mathbb{C}$ be functions on S. Verify that the sequence $(f_n)_n$ converges uniformly on S to f if and only if the sequence of sup-norms $\|f_n - f\|_S$ converges to 0.

4. (a) Prove that $f(z) = \sum_{n=0}^{\infty} nz^n$ is continuous in the disc $|z| < 1$.

(b) Prove that $g(z) = \sum_{n=1}^{\infty} \frac{1}{n^2+z}$ is continuous in the right half-plane $\mathrm{Re}(z) > 0$.

5. Check that:

(a) $e^{z_1} = e^{z_2} \iff \frac{z_1 - z_2}{2\pi i} \in \mathbb{Z}$

(b) $e^{z+2k\pi i} = e^z$ for all $z \in \mathbb{C}$ and all $k \in \mathbb{Z}$.

6. (a) Find the image of the set $\{z \in \mathbb{C} : \mathrm{Im}(z) = \pi/4\}$ by the exponential function $\exp(z)$. In general, what are the images by $\exp(z)$ of horizontal lines?

(b) Find the image of the set $\{z \in \mathbb{C} : \mathrm{Re}(z) = 0\}$ by the exponential function $\exp(z)$. In general, what are the images by $\exp(z)$ of vertical lines?

7. Check that $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$ for all $z \in \mathbb{C}$.

8. Use the Euler formulas to prove that:

(a) $\cos(z) = 0 \;\Leftrightarrow\; z = (2k+1)\frac{\pi}{2},\; k \in \mathbb{Z}$

(b) $\sin(z) = 0 \;\Leftrightarrow\; z = k\pi,\; k \in \mathbb{Z}$.

That is, show that the complex sine and cosine functions vanish only on the zeros of their real counterparts.

9. Check the following set-theoretical equalities

(a) $\log(zw) = \log z + \log w$ for all $z, w \in \mathbb{C}$

(b) $\log(z/w) = \log z - \log w$ for all $z, w \in \mathbb{C}$, $w \neq 0$,

where $A \pm B = \{a \pm b : a \in A, b \in B\}$ for subsets $A, B \subset \mathbb{C}$.