Problem Set 4
September 27, 2025.

1. Sketch the set $\left\{z \in \mathbb{C} : \dfrac{|e^z - 1|}{|e^z + 1|} = 1\right\}$. Justify your answer.

2. For each of the following series, find the radius of convergence R, and discuss convergence on the circle $C(0; R)$:

(a) $\displaystyle\sum_0^\infty n^2 z^n$ (b) $\displaystyle\sum_0^\infty \frac{z^{2n}}{4^n}$ (c) $\displaystyle\sum_0^\infty n! z^n$

(d) $\displaystyle\sum_0^\infty \frac{z^n}{1+2^n}$ (e) $\displaystyle\sum_1^\infty \frac{n! z^n}{n^n}$ (f) $\displaystyle\sum_1^\infty \frac{2^n z^n}{n!}$.

3. Suppose the series $\sum_{n=0}^\infty a_n z^n$ has radius of convergence R. Determine the radius of convergence of:

(a) $\sum_{n=0}^\infty n^k a_n z^n$, where $k \in \mathbb{Z}_+$ is a constant
(b) $\sum_{n=0}^\infty |a_n| z^n$
(c) $\sum_{n=0}^\infty a_n^2 z^n$.

4. Prove that the following function series converge absolutely and uniformly on the open unit disc $D(0; 1)$:

(a) $\sum_1^\infty \frac{z^{n+1}}{n!}$
(b) $\sum_1^\infty \frac{z^{2n}}{n^2}$
(c) $\sum_1^\infty \left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots + \frac{z^n}{2^n} - \frac{2}{2-z}\right)$.

5. Prove that the entire functions exp, sin, and cos are the unique extensions to $\mathbb{C}$ of the functions of real variable exp, sin, and cos, respectively. [Hint: Suppose $f, g \in \mathcal{O}(\mathbb{C})$ are such that both $f|_\mathbb{R}$ and $g|_\mathbb{R}$ coincide with the natural exponential function $\exp : \mathbb{R} \to \mathbb{R}$. Pick any $z_0 \in \mathbb{R} \subset \mathbb{C}$ and observe that z_0 is a limit point of the set $\mathbb{R}$.]

6. Let $D \subset \mathbb{C}$ be a non-empty domain. Prove that the ring of analytic functions $\mathcal{O}(D)$ is an integral domain. That is, show that if two functions $f, g \in \mathcal{O}(D)$ satisfy $f \cdot g(z) = 0$ for all $z \in D$, then at least one of the functions is identically zero on D.

7. For each of the following functions, find its domain and the largest set on which the function is analytic:

(a) $e^z \cos z$
(b) $\frac{e^z}{(z-i)^{100}}$
(c) $\frac{(1-z)\sin z}{(z^4-1)^8}$
(d) $\sum_{n=0}^\infty n z^n$.

8. Show that $f(z) = \bar{z}$ is not $\mathbb{C}$-differentiable at any point in $\mathbb{C}$.