Problem Set 5
October 4, 2025.

1. Find first five non-zero terms of the power series representation of each of the following functions at the given center:

(a) $\dfrac{\sin z}{z},\ z_0 = 1,$ (b) $z^2 e^z,\ z_0 = 0,$ (c) $e^z \sin z,\ z_0 = \pi.$

2. (a) Use the power series definition of the exponential function (and the Term-by-Term Differentiation Theorem proved in class) to show that $(e^z)' = e^z$, for all $z \in \mathbb{C}$.

(b) Use the definition of complex functions sin and cos (or Euler formulas) to show that $(\sin z)' = \cos z$ and $(\cos z)' = -\sin z$, for all $z \in \mathbb{C}$.

3. Determine which of the following polynomials are $\mathbb{C}$-differentiable:

(a) $P(x+iy) = x^3 - 3xy^2 - x + i(3x^2y - y^3 - y)$
(b) $P(x+iy) = x^2 + iy^2$
(c) $P(x+iy) = 2xy + i(y^2 - x^2)$.

4. Show that there is no power series $f(z) = \sum a_n z^n$ with positive radius of convergence and such that $f(1/k) = 1$ for all $k \in \mathbb{Z}_+$, and $f'(0) \neq 0$.

5. (a) Find all entire functions $f = u + iv$ with $u(x+iy) = x^2 - y^2$.

(b) Show that there are no analytic functions $f = u + iv$ with $u(x+iy) = x^2 + y^2$.

6. For each of the following functions, show it is harmonic on a domain D (what's the largest such D?) and find its harmonic conjugate:

(a) $u(x,y) = e^x \cos y$
(b) $u(x,y) = \ln \sqrt{x^2 + y^2}$
(c) $u(x,y) = \sin x \cdot \dfrac{(e^y + e^{-y})}{2}$.

7. Let f be a non-constant function analytic in the disc $D(0;2)$ and such that $f(z) \neq 0$ for all z with $|z| \leq 1$. Prove that there is no point $z_0 \in D(0;1)$ satisfying $|f(z_0)| \leq |f(z)|$ for all z with $|z| = 1$.

8. Evaluate $\displaystyle\int_\gamma z^2\, dz$, where γ is the curve $\gamma(t) = t^2 + it^3$, $t \in [0,1]$.

9. Evaluate $\displaystyle\int_\gamma \frac{dz}{z}$, where γ is given as $\gamma(t) = \sin t + i \cos t$, $t \in [0, 2\pi]$. Compare the answer with the formula given in class for integrals of $(z - z_0)^n$ along a circle centered at z_0 and explain the difference.

10. Find a real number $M > 0$ such that $\left| \displaystyle\int_\gamma \frac{dz}{2 + z^2} \right| \leq M$, where γ is the upper half of the unit circle.