Problem Set 6
October 18, 2025.

1. Let C be the arc of the circle $|z| = 2$ that lies in the first quadrant. Show that $\left|\int_C \frac{dz}{z^2+1}\right| \leq \frac{\pi}{3}$.

2. Evaluate the integrals

(a) $\int_{\gamma_1} \bar{z}\, dz$

(b) $\int_{\gamma_2} \bar{z}\, dz$

(c) $\int_{\gamma_1} \frac{e^z}{(z+i)^2}\, dz$

(d) $\int_{\gamma_2} \frac{e^z}{(z+i)^2}\, dz$,

where γ_1 is the interval $[0,1]$ and $\gamma_2 = [0,i] \cup [i, i+1] \cup [i+1, 1]$ is the remaining part of the circumference of a unit square centered at $1/2 + i/2$.

3. Suppose that an analytic function f agrees with $\tan x$ for $x \in [0,1]$. Show that $f(z) = i$ has no solution. Could f be entire?

4. Find the maximum and minimum moduli of $z^2 - z$ for $|z| \leq 2025$. Justify your answer.

5. Suppose that f is analytic in the closed unit disc $\overline{D(0,1)}$ (i.e., by definition, $f \in \mathcal{O}(\Omega)$ for some open $\Omega \subset \mathbb{C}$ containing $\overline{D(0,1)}$). Prove that then there exists $n \in \mathbb{Z}_+$ such that $f(1/n) \neq 1/(n+1)$. [Hint: Express $1/(n+1)$ in terms of $1/n$.]