# Problem Set 7

November 8, 2025.

**1.** Evaluate the following integrals. Justify your answers.

(a) $\displaystyle\int_{C(1;1)} \frac{dz}{z^2-1}$ (b) $\displaystyle\int_{C(1;1)} \frac{e^z\ dz}{(z^2-1)^2}$

(c) $\displaystyle\int_{C(1;\frac{1}{2})} \frac{dz}{(1-z)^3}$ (d) $\displaystyle\int_{C(i;\frac{1}{2})} \frac{z^3\cos z}{z^4-1}\,dz.$

**2.** Prove that there is no non-constant entire function $f$ satisfying $f(z+1) = f(z)$ and $f(z-2i) = f(z)$ for all $z \in \mathbb{C}$. [Hint: Show that such an $f$ would have to be bounded.]

**3.** Show that every polynomial $P(z) \in \mathbb{C}[z]$ with *real* coefficients and of odd degree has a real root.

**4.** Use Problem 3 to prove that every non-constant polynomial with real coefficients is a product of linear and quadratic polynomials with real coefficients.

**5.** Let $f$ be an entire function. Prove that $f$ is a polynomial if and only if there exist $R > 0$, $C > 0$ and $N \in \mathbb{Z}_+$ such that $|f(z)| \leq C|z|^N$ for all $z$ with $|z| \geq R$. [Hint: Use the Cauchy Estimates theorem.]

**6.** Suppose that $f$ is an entire function and there exist $R > 0$ and $N \in \mathbb{Z}_+$ such that $|f(z)| \geq |z|^N$ for $|z| \geq R$. Prove that $f$ is a polynomial of degree greater than or equal to $N$. [Hint: First show that $f$ has only finitely many (not necessarily distinct) zeros in $\mathbb{C}$; say, $\zeta_1, \ldots, \zeta_s$. Next analyze the function $g(z) = f(z)/((z-\zeta_1)\ldots(z-\zeta_s))$.]

**7.** The *Open Mapping Theorem* states that the image of an open set under a non-constant analytic function is an open set. Prove the theorem, by following these steps:

(a) Use the Maximum Modulus Principle to prove the following statement (called the *Minimum Modulus Principle*): If $f$ is a non-constant analytic function in a domain $D$, then for every $z_0 \in D$, $|f(z_0)|$ is not a local minimum of $|f(z)|$, unless $f(z_0) = 0$. (Equivalently, if $f(z) \neq 0$ for all $z \in D$, then $|f(z)|$ attains its minimum on the boundary $\partial D$.)

(b) Suppose $S$ is an open set, $f \in \mathcal{O}(S)$ is non-constant, $z_0 \in S$, and $f(z_0) = 0$. Show that there is $r > 0$ such that $\{z : |z - z_0| \leq r\} \subset S$ and $f(z) \neq 0$ for all $z$ with $|z - z_0| = r$.

(c) For $r > 0$ as above, let $C$ denote the circle $\{z : |z - z_0| = r\}$ and set $\varepsilon := 1/2 \cdot \min\{|f(z)| : z \in C\}$. Prove that $f(D(z_0; r))$ contains the open disc $D(0; \epsilon)$: Given $w \in \mathbb{C}$ with $|w| < \varepsilon$, for all $z \in C$ we have

$$|f(z) - w| \geq |f(z)| - |w| \geq \varepsilon\,.$$

On the other hand, $|f(z_0)-w| = |0-w| = |w| < \varepsilon$. Therefore, the global minimum of the continuous function $|f(\cdot) - w|$ on the closed set $\overline{D(z_0;r)}$ (which exists by the Extreme Value Thm.) must be attained at some point $z' \in D(z_0;r)$. By (a), this minimum must be equal to zero (as $f$ is non-constant), hence $f(z') = w$.

(d) Suppose $f(z_0) \neq 0$. Define $g(z) = f(z) - f(z_0)$ and repeat the argument of (c) for $g$ to show that $f(D(z_0;r))$ contains $D(f(z_0);\varepsilon)$.

(e) Use part (d) to conclude the theorem.

**8.** Suppose $f \in \mathcal{O}(S)$ is non-constant and $f(S) = T$. Use Problem 7, to prove that if $f(z_0)$ is a boundary point of $T$ then $z_0$ is a boundary point of $S$.

**9.** Suppose $f$ is analytic in some open $\Omega \subset \mathbb{C}$ containing the closed unit disc $\overline{D(0;1)}$ and maps the unit circle $C(0;1)$ into itself. Prove that $f$ maps the open unit disc $D(0;1)$ *onto* itself. [Hint: First, use the Maximum Modulus Principle to show that $f$ maps $D(0;1)$ *into* itself. Then use Problem 8, to show that $f(D(0;1)) \supset D(0;1)$.]