Problem Set 8

November 15, 2025.

1. Prove that the image of $\mathbb{C}$ by a non-constant entire function is dense in $\mathbb{C}$.

[Hint: Consider separately the cases when f is polynomial and transcendental (non-polynomial).]

2. For each of the following functions, find its Laurent expansion convergent in some punctured disc centered at z_0, $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$:

(a) $f(z) = \dfrac{1}{z^4 + z^2}$, $z_0 = 0$

(b) $f(z) = \dfrac{e^{1/z^2}}{z}$, $z_0 = 0$

(c) $f(z) = \dfrac{1}{z^2 - 4}$, $z_0 = 2$.

3. For each of the functions in Problem 2, find the integrals

$$\int_{C(z_0;\varepsilon)} f(\zeta)d\zeta \quad \text{and} \quad \int_{C(z_0;\varepsilon)} f(\zeta)(\zeta - z_0)^{2025} d\zeta\,,$$

where ε is some small positive constant.

[Hint: Do not attempt a direct computation.]

4. Find *all* different Laurent expansions about $z_0 = 1$ of the function

$$f(z) = \frac{1}{z} - \frac{2z}{z-2} + \frac{z^3}{z+3}\,,$$

and determine their respective annuli of convergence.

5. Evaluate $\displaystyle\int_{C(0;2)} ze^{3/z} dz$. Justify your answer.