Problem Set 9

November 22, 2025.

1. (a) Prove that every non-constant entire function has either a pole or an essential singularity at infinity.

(b) Let $f \in \mathcal{O}(\mathbb{C})$ and suppose that $\lim_{z\to\infty} f(z) = \infty$. Prove that f is a polynomial.

2. Find all (up to continuous deformation) piecewise smooth simple closed curves γ for which the following equality holds:

$$\int_\gamma \frac{-z^3}{z^2+z+1}\,dz \;=\; \int_\gamma \frac{\sin z}{z^2}\,dz\;,$$

and evaluate the common integral value in these cases. Justify.

3. Derive the Fundamental Theorem of Algebra as a corollary of Rouché's Theorem.

4. Let $D \subset \mathbb{C}$ be a bounded domain, and let $f \in \mathcal{O}(\overline{D})$. Show that if f is non-constant, then $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ assume their maxima and minima on the boundary ∂D. [Hint: Use the Open Mapping Theorem.]

5. Show that there is no non-constant function $f \in \mathcal{O}(\overline{D(0;1)})$, which is real-valued on the unit circle.

6. For each of the following functions, find the number of zeros in the given region. Justify your answer.

(a) $f(z) = 3e^z - z$ in $|z| \le 1$

(b) $f(z) = e^z/3 - z$ in $|z| \le 1$

(c) $f(z) = z^4 - 5z + 1$ in $1 \le |z| \le 2$

(d) $f(z) = z^6 - 5z^4 + 3z^2 - 1$ in $|z| \le 1$.

7. Suppose that an entire function f satisfies inequalities $0 < |f(z)| < \dfrac{2}{e^2}$ whenever $|z| = 2$. Prove that the function $\dfrac{f(z)}{e^{iz}}$ has precisely one fixed point in the open disc $\{z \in \mathbb{C} : |z| < 2\}$.