Practice Midterm I

0. Problems from Problem Sets 1–5.

1. (a) State definitions of the argument and the principal argument of a non-zero complex number.
(b) State the definition of the logarithm of a complex number.
(c) Find the set of those $z \in \mathbb{C} \setminus \{0\}$ for which $\operatorname{Arg}(\bar{z}) = \operatorname{Arg}(z^{-1})$. Justify your answer.

2. Find i^i. Justify your answer.

3. Sketch the set $\left\{z \in \mathbb{C} : \frac{|e^z+2025|}{|e^z-2025|} = 1\right\}$. Justify your answer.

4. (a) State one of the equivalent definitions of $\mathbb{C}$-differentiability of a function of complex variable.
(b) State the Cauchy-Riemann equations (and explain your notation).
(c) Prove that the function $f(z) = e^{\bar{z}}$ is not $\mathbb{C}$-differentiable at any point in $\mathbb{C}$.

5. State the definition of an analytic function and find the domain of analyticity of $f(z) = \frac{1}{z-2} - \frac{e^z}{2z}$. What is the radius of convergence of the power series representation of f at $z_0 = 2i$?

6. Let $\Omega = \{z \in \mathbb{C} : \operatorname{Re}(z) < 0\}$. Prove that the series $\sum_1^\infty \frac{1}{z-n^2}$ converges for every $z \in \Omega$. Prove that the function $f : \Omega \to \mathbb{C}$ defined as $f(z) = \sum_1^\infty \frac{1}{z-n^2}$ is continuous in Ω.

7. Find the sum of the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{(n^2-n)\pi^n}{2^{2n}}$.

8. For each of the following functions $v(x,y)$, find its harmonic conjugate in the right half-plane (if exists) and find a closed form of a function $f(z)$ whose *imaginary* part equals v:

(a) $v(x,y) = \arctan\frac{y}{x}$
(b) $v(x,y) = 6x^2y - 2y^3$
(c) $v(x,y) = e^x \cos(iy)$.

9. (a) State the definitions of a harmonic function and its harmonic conjugate.
(b) Find all harmonic conjugates of $v(x,y) = \dfrac{\sin x}{e^y}$.
(c) Find an entire function f (in its closed form) whose imaginary part equals v.

10. Show that if $f = u + iv$ is $\mathbb{C}$-differentiable in an open set Ω and u is constant on Ω, then so is f.

11. Find the maximum of $|ze^{z^2-1}|$ on the closed unit disc.

12. Use the definition of contour integral to prove that

(a) $\int_\gamma af + bg \ = \ a\int f + b\int g$
(b) $\int_{-\gamma} f = -\int_\gamma f$
(c) $\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$,

where $a, b \in \mathbb{C}$, f, g are continuous functions, and $\gamma, \gamma_1, \gamma_2$ are piecewise $\mathcal{C}^1$ curves.