Practice Midterm II

0. Problems from Problem Sets 6 and 7.

1. Review the Characterization of Analytic Functions Theorem (a.k.a. the 7 Equivalent Conditions Thm.) and give a detailed proof.

2. (a) State the Cauchy Integral Formula in the maximal generality you know to hold true.

(b) Give one example for necessity of each of the assumptions made in your statement of *CIF*.

3. (a) Let f be analytic in a simply connected domain $D \subset \mathbb{C}$, and let γ be a piecewise C^1 non-degenerate closed curve in D, such that $f(\zeta) = 0$ for all $\zeta \in \gamma$. Prove that f is identically zero on D.

(b) Give an example of a *non*-simply connected open set $D \subset \mathbb{C}$, a function $f \in \mathcal{O}(D)$, and a simple closed curve $\gamma \in D$, such that $\int_\gamma f(\zeta)\, d\zeta \neq 0$. Justify your answer.

4. Let f be analytic on the unit disc, and assume that $|f(z)| \leq 1$ for $|z| = 1/2$. Prove that $|f'(0)| \leq 2$.

5. (a) State the definition of index (winding number) of a piecewise $\mathcal{C}^1$ closed curve with respect to a point.

(b) Sketch a closed curve γ for which there exist points z_0, z_1, z_2 with $I(\gamma; z_j) = j$, for $j = 0, 1, 2$. (Identify those points in your picture.)

(c) Let γ and $z_2 \in \mathbb{C}$ be as in part (b). Explain why $\int_\gamma \frac{d\zeta}{\zeta - z_2}$ is not equal to $\int_{C(z_2;\varepsilon)} \frac{d\zeta}{\zeta - z_2}$.

6. (a) Give an example of a bounded domain Ω in $\mathbb{C}$ and a piecewise $\mathcal{C}^1$ closed curve γ in Ω such that $I(\gamma; a) = 5$ for some point $a \in \mathbb{C} \setminus \Omega$.

(b) Give an example of a bounded domain Ω in $\mathbb{C}$ and a cycle $\Gamma = \gamma_1 + \cdots + \gamma_s$ (for some $s \in \mathbb{Z}_+$) such that each γ_j is a $\mathcal{C}^1$ *simple* closed curve in Ω, no two γ_i, γ_j intersect, and for every $k \in \{1, \ldots, 5\}$ there is a point $a_k \in \mathbb{C} \setminus \Omega$ such that $I(\Gamma; a_k) = k$.

7. (a) State the definition of a piecewise $\mathcal{C}^1$ simple closed curve in $\mathbb{C}$.

(b) Give an example of a piecewise $\mathcal{C}^1$ simple closed curve γ for which

$$\int_\gamma \frac{z^3 e^{1/z}}{(z^2 + z + 1)(z^2 + 1)}\, dz \; = \; 0\,.$$

Justify your answer.

(c) Let γ be the circle $C(0; 2)$ traversed (one time) counter-clockwise. Prove that there exist $k \in \mathbb{Z}_+$ and $\mathcal{C}^1$ simple closed curves $\gamma_1, \ldots, \gamma_k$ such that the cycle $\Gamma := \gamma - \sum_{j=1}^k \gamma_j$ satisfies

$$\int_\Gamma \frac{z^3 e^{1/z}}{(z^2 + z + 1)(z^2 + 1)}\, dz \; = \; 0\,.$$

8. Find a real number $M \geq 0$ such that $\left| \int_{C(0;1)} \frac{2z + 1}{z^5}\, dz \right| \leq M$. What is the least such M? Justify your answer.

9. Let γ denote the circle $C(0;2)$ traversed two times counter-clockwise. Evaluate the following integrals. [Hint: Use the Cauchy-Dixon Theorem to represent a given integral as a linear combination of certain simpler integrals.]

$$\text{(a)} \int_\gamma \frac{\sin(\pi z)}{z^3-1}\,dz \qquad \text{(b)} \int_\gamma \frac{e^z}{(z^2+1)^2}\,dz \qquad \text{(c)} \int_\gamma \frac{dz}{(z^2+2z-3)^4}\,.$$

10. Let γ denote the limaçon $r = 2 + 4\cos\theta$ ($\theta \in [0, 2\pi]$) traversed three times counter-clockwise.

(a) What is the index of γ with respect to $z_0 = 1$?

(b) Evaluate the integral

$$\int_\gamma \frac{z^5 + 2z^4 + 3z^3 + 4z^2 + 5z}{z^2-1}\,dz\,.$$

11. Find and classify all singularities in the Riemann sphere $\widehat{\mathbb{C}}$ of the following functions. Justify your answer.

$$\text{(a) } f(z) = \frac{e^z}{z^4+z^2} \qquad \text{(b) } f(z) = \frac{\sin(\pi z)}{z^3(z^2+1)} \qquad \text{(c) } f(z) = \frac{\sin(z^2)}{z^4+z^2} \qquad \text{(d) } f(z) = \frac{e^{1/z}}{z^2+1}\,.$$

12. (a) How many different Laurent expansions centered at $z_0 = i$ are there for the function $f(z) = \frac{1}{z^2+1} - \frac{2}{z+1}$? Justify your answer.

(b) Find all the Laurent series of $f(x)$, and for each one determine the largest region in which it is convergent.