The University of Western Ontario
Department of Mathematics

Mathematics 3124A/9024A, Fall 2018: COMPLEX ANALYSIS I

Final Exam Review

Material

- Harmonic functions - Laplace's equation, finding harmonic conjugates, existence of analytic function with the prescribed real/imaginary part.
- Power series - definition and calculation of radius of convergence.
- Abel-Weierstrass Lemma, Identity Principle for power series and analytic functions.
- Theorems on term by term differentiation / integration; evaluation of infinite sums by means of these theorems.
- Definition of a line integral.
- Equivalent conditions for vanishing of a closed contour integral.
- Equivalent conditions for analyticity of a function; determining the domain of analyticity by means of the various equivalent conditions, like CR-equations (e.g., $\bar{z}$, $|z|$, $\operatorname{Im} z$, etc.), convergence of Taylor expansion (e.g., $\sum \frac{z^n}{2}$, $\frac{1}{z-a}$, etc.), Cauchy Thm. for triangles + Morera Theorem.
- Index (winding number) of a curve with respect to a point.
- Cauchy Closed Curve Theorem and Cauchy Integral Formula in full generality (Cauchy-Dixon).
- Consequences of Cauchy Theorem for entire functions: Fundamental Theorem of Algebra, Liouville's Theorem.
- Other consequences: Cauchy Estimates, Maximum Modulus Principle, Open Mapping Theorem, Schwarz Lemma, Schwarz Reflection Principle.
- Definition and classification of isolated singularities, equivalent characterizations.
- Laurent expansion - existence of, and radii of convergence; explicit evaluations using Geometric Series Theorem; integral representation of coefficients in the expansion.
- Definition(s) of residue; explicit calculation of residues by means of Cauchy Integral Formula or Laurent expansion.
- Singularities at (complex) infinity.
- Residue Theorem in full generality, Rouché's Theorem.
- Evaluation of definite integrals by means of Residue Theorem.

Practice problems

1. Problems from Problem Sets 1–9.

2. If $u(x, y)$ and $v(x, y)$ are harmonic, are the following functions harmonic?

   (a) $u(v(x, y), 0)$

   (b) $u(x, y) \cdot v(x, y)$

   (c) $u(x, y) + v(x, y)$.

3. Let γ be the unit circle. Prove that $\left| \int_\gamma \frac{\sin z}{z^2}\, dz \right| \leq 2\pi e$.

4. Show that $\sum_{n=1}^{\infty} \frac{1}{z^n}$ is analytic on $A = \{z \in \mathbb{C} : |z| > 1\}$.

5. Show that $\sum_{n=1}^{\infty} \frac{1}{n! z^n}$ is analytic on $\mathbb{C} \setminus \{0\}$. Evaluate its integral around the unit circle.

6. Evaluate the sums:

   (a) $\sum_{n=1}^{\infty} \frac{n\pi^n}{(2e)^n}$

   (b) $\sum_{n=1}^{\infty} \frac{(-1)^n \pi^{n+1}}{2^n (2n-1)!}$.

7. Characterize the piecewise $\mathcal{C}^1$ simple closed curves γ for which the following equation holds

$$\int_\gamma \frac{z^3 e^{1/z}}{(z^2 + z + 1)(z^2 + 1)}\, dz \;=\; 0\,.$$

8. Let f be a function analytic in $\mathbb{C}$ except for a finite number of singularities $\{z_1, \ldots, z_s\}$, and let $R > \max\{|z_j| : j = 1, \ldots, s\}$. Prove that, if $|f(z)| \leq M$ for all $z \in C(0, R)$ and $\lim_{z\to\infty} f(z) = 0$, then $|f(z)| \leq M$ for all z with $|z| \geq R$.

9. Let f be a function analytic in the open upper half plane, continuous on the closed upper half plane, and real on the real axis. Prove that, if $|f(z)| \leq 1$ for $z \in \{x + iy \in \mathbb{C} : y \geq 0, x^2 + y^2 = 4\}$ and $\lim_{z\to\infty} f(z) = 0$, then f is a constant function.

10. Show that there is exactly one point z in the right half plane $\{w : \mathrm{Re}(w) > 0\}$, at which $z + e^{-z} = 2$. [Hint: For R big enough, consider the curve given as the boundary of a right semicircle with radius R centered at the origin.]

11. Let f be analytic in a domain containing the closed unit disc, and such that $0 < |f(z)| < 1$ for $|z| = 1$. Show that f has exactly one fixed point z_0 (i.e., $f(z_0) = z_0$) inside the unit disc.

12. Evaluate the integrals:

    (a) $\int_{-\infty}^{\infty} \frac{dx}{1 + x^6}$

    (b) $\int_0^{\infty} \frac{x \sin x}{1 + x^3}\, dx$

    (c) $\int_{-\infty}^{\infty} \frac{dx}{x(x^3 - 1)}$

(d) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta}$.

13. Classify the singularity at infinity, and evaluate the residue at infinity of the following functions:

(a) $f(z) = \dfrac{z^6}{6z^5 - 4z^3 + 2z}$

(b) $f(z) = \dfrac{\sin z}{4z^3 + 2z}$

(c) $f(z) = \dfrac{z^4}{6z^5 - 4z^3 + 2z}$

(d) $f(z) = \dfrac{\sin(e^{\cos z})}{3z^2}$.