Problem Set 2
September 27, 2018.

1. Check the following set-theoretical equalities

(a) $\log(zw) = \log z + \log w$, for all $z, w \in \mathbb{C}$
(b) $\log(z/w) = \log z - \log w$, for all $z, w \in \mathbb{C}$, $w \neq 0$,

where $A \pm B = \{a \pm b : a \in A, b \in B\}$ for subsets $A, B \subset \mathbb{C}$.

2. Prove that every open connected set S in $\mathbb{C}$ is path-connected.

[Hint: Given $z_0 \in S$, consider the set S_0 of points in S that can be connected by a path with z_0. Show that S_0 is non-empty, open and closed. Use definition of connectedness to derive that $S \setminus S_0 = \varnothing$.]

3. Let $P(z) = a_0 z^n + \cdots + a_{n-1}z + a_n$ be a polynomial with complex coefficients, where $a_0 \neq 0$, $n \in \mathbb{Z}_+$, and let $R = \max\{\sqrt[k]{\dfrac{|a_k|}{|a_0|}} : k = 1, \ldots, n\}$. Prove that, if $P(z_0) = 0$ then $|z_0| \leq 2R$.

4. Find the 5'th roots of the number $-1 + i\sqrt{3}$.

5. For $n \geq 3$, let D_n be a regular n-gon inscribed in the unit circle. Suppose that one of the vertices of D_n is at 1, and let $d_1, d_2, \ldots, d_{n-1}$ be the distances of the other vertices from 1. Prove that $d_1 d_2 \ldots d_{n-1} = n$.

6. Prove that $g(z) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + z}$ is continuous in the right half-plane $\{z : \mathrm{Re}(z) > 0\}$.

7. Prove that the following function series converge absolutely and uniformly on the open unit disc $D(0, 1)$:

(a) $\sum_1^\infty \frac{z^{n+1}}{n!}$
(b) $\sum_1^\infty \frac{z^{2n}}{n^2}$
(c) $\sum_0^\infty \left(\frac{z}{2}\right)^n$
(d) $\sum_1^\infty \left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots + \frac{z^n}{2^n} - \frac{2}{2-z}\right)$.

8. For each of the following series, find the radius of convergence R, and discuss convergence on the circle $C(0, R)$:

(a) $\sum_0^\infty n^2 z^n$
(b) $\sum_0^\infty n! z^n$
(c) $\sum_0^\infty \frac{z^n}{1+2^n}$
(d) $\sum_1^\infty \frac{n! z^n}{n^n}$
(e) $\sum_1^\infty \frac{2^n z^n}{n!}$.