Problem Set 3

October 20, 2018.

1. For each of the following functions, find its domain and the largest set on which the function is analytic:

(a) $e^z \cos z$

(b) $\frac{e^z}{(z-i)^{100}}$

(c) $\frac{(1-z)\sin z}{z^4-1}$

(d) $\sum_{n=0}^{\infty} nz^n$.

2. Show that $f(z) = \bar{z}$ is not $\mathbb{C}$-differentiable at any point in $\mathbb{C}$.

3. Determine which of the following polynomials are $\mathbb{C}$-differentiable:

(a) $P(x+iy) = x^3 - 3xy^2 - x + i(3x^2y - y^3 - y)$

(b) $P(x+iy) = x^2 + iy^2$

(c) $P(x+iy) = 2xy + i(y^2 - x^2)$.

4. Show that there is no power series $f(z) = \sum a_n z^n$ with positive radius of convergence and such that
(i) $f(z) = 1$ for $z = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ and
(ii) $f'(0) > 0$.

5. Show that there are no analytic functions $f = u + iv$ with $u(x+iy) = x^2 + y^2$.

6. Suppose D is a non-empty domain in $\mathbb{C}$, $f \in \mathcal{O}(D)$, and a sequence $(z_k)_{k=1}^{\infty}$ converges to a point $z_0 \in D$. Prove that, if $f'(z_k) = 0$ for all $k \in \mathbb{Z}_+$, then f is a constant function.

7. Find harmonic conjugates for the following functions:

(a) $u(x,y) = e^x \cos y$

(b) $u(x,y) = \ln \sqrt{x^2 + y^2}$

(c) $u(x,y) = \sin x \cdot \dfrac{(e^y + e^{-y})}{2}$.