Problem Set 4
October 26, 2018.

1. Let f be a non-constant function analytic in the disc $D(0,2)$ and such that $f(z) \neq 0$ for all z with $|z| \leq 1$. Prove that there is no point $z_0 \in D(0,1)$ satisfying $|f(z_0)| \leq |f(z)|$ for all z with $|z| = 1$.

2. Use the definition of line integral to prove that

(a) $\int_\gamma \alpha f + \beta g \; = \; \alpha \int_\gamma f + \beta \int_\gamma g$

(b) $\int_{-\gamma} f = -\int_\gamma f$

(c) $\int_{\gamma_1 + \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$,

where $\alpha, \beta \in \mathbb{C}$, f, g are continuous functions, and $\gamma, \gamma_1, \gamma_2$ are piecewise $\mathcal{C}^1$ curves (such that $\gamma_1 + \gamma_2$ is well-defined).

3. Evaluate $\displaystyle\int_\gamma z^2\, dz$, where γ is the curve $\gamma(t) = t^2 + it^3$, $t \in [0,1]$.

4. Evaluate $\displaystyle\int_\gamma \frac{dz}{z}$, where γ is given as $\gamma(t) = \sin t + i\cos t$, $t \in [0, 2\pi]$. Compare the answer with the formula given in class for integrals of $(z - z_0)^n$ along a circle centered at z_0 and explain the difference.

5. Find a real number $M > 0$ such that $\displaystyle\left|\int_\gamma \frac{dz}{2+z^2}\right| \leq M$, where γ is the upper half of the unit circle.

6. Let C be the arc of the circle $\{z : |z| = 2\}$ that lies in the first quadrant. Show that $\displaystyle\left|\int_C \frac{dz}{z^2+1}\right| \leq \frac{\pi}{3}$.

7. Evaluate the integrals

(a) $\int_{\gamma_1} \bar{z}\, dz$

(b) $\int_{\gamma_2} \bar{z}\, dz$

(c) $\int_{\gamma_1} \frac{e^z}{(z+i)^2}\, dz$

(d) $\int_{\gamma_2} \frac{e^z}{(z+i)^2}\, dz$,

where γ_1 is the interval $[0,1]$ and $\gamma_2 = [0, i] \cup [i, i+1] \cup [i+1, 1]$ is the remaining part of the circumference of a unit square centered at $1/2 + i/2$ (traversed clockwise).