Problem Set 5

November 2, 2018.

1. Let $D \subset \mathbb{C}$ be a nonempty domain (i.e., a connected open set). Prove that for every $a, b \in D$ there exists a piecewise $\mathcal{C}^1$ curve $\gamma : [0,1] \to D$ satisfying $\gamma(0) = a$ and $\gamma(1) = b$.
[Hint: Fix $z_0 \in D$ and let S be the set of all points $z \in D$ which can be connected with z_0 by a piecewise $\mathcal{C}^1$ curve in D. Show that S is nonempty, open and closed.]

2. Find the maximum and minimum moduli of $z^2 - z$ for $|z| \leq 2018$. Justify your answer.

3. Find the maximum of $|ze^{z^2-1}|$ on the closed unit disc.

4. Evaluate the following integrals. Justify your answers.

(a) $\displaystyle\int_{C(1,1)} \frac{dz}{z^2-1}$

(b) $\displaystyle\int_{C(1,1)} \frac{e^z\, dz}{(z^2-1)^2}$

(c) $\displaystyle\int_{C(1,\frac{1}{2})} \frac{dz}{(1-z)^3}$

(d) $\displaystyle\int_{C(i,\frac{1}{2})} \frac{z^3 \cos z}{z^4-1}\, dz\,.$

5. Suppose that an analytic function f (defined on some open $\Omega \subset \mathbb{C}$ containing the interval $[0,1] \subset \mathbb{R}$) agrees with $\tan x$ for $x \in [0,1]$. Show that $f(z) = i$ has no solution.

6. Suppose that f is analytic in the closed unit disc $\overline{D(0,1)}$ (i.e., by definition, $f \in \mathcal{O}(\Omega)$ for some open $\Omega \subset \mathbb{C}$ containing $\overline{D(0,1)}$). Prove that then there exists $n \in \mathbb{Z}_+$ such that $f\left(\frac{1}{n}\right) \neq \frac{1}{n+1}$. [Hint: Suppose otherwise. Express $1/(n+1)$ in terms of $1/n$ to show that f coincides with another function g at every $z_n = 1/n$. Use the Local Isolation of Zeros to show that $f = g$. Finally, get a contradiction by showing that g is not analytic in the whole $\overline{D(0,1)}$.]