Problem Set 6

November 9, 2018.

1. Prove that there is no non-constant entire function f satisfying $f(z+1) = f(z)$ and $f(z-2i) = f(z)$ for all $z \in \mathbb{C}$. [Hint: Show that such an f would have to be bounded.]

2. Suppose that f is an entire function and there exist $R > 0$ and $N \in \mathbb{Z}_+$ such that $|f(z)| \geq |z|^N$ for $|z| \geq R$. Prove that f is a polynomial of degree greater than or equal to N. [Hint: First show that f has only finitely many (not necessarily distinct) zeros in $\mathbb{C}$; say, $\zeta_1, \ldots, \zeta_s$. Next analyze the function $g(z) = f(z)/((z-\zeta_1)\ldots(z-\zeta_s))$.]

3. The *Open Mapping Theorem* states that the image of an open set under a non-constant analytic function is an open set. Prove the theorem, by following these steps:

(a) Use the Maximum Modulus Principle to prove the following statement (called the *Minimum Modulus Principle*): If f is a non-constant analytic function in a domain D, then for every $z_0 \in D$, $|f(z_0)|$ is not a local minimum of $|f(z)|$, unless $f(z_0) = 0$. (Equivalently, if $f(z) \neq 0$ for all $z \in D$, then $|f(z)|$ attains its minimum on the boundary ∂D.)

(b) Suppose S is an open set, $f \in \mathcal{O}(S)$ is non-constant, $z_0 \in S$, and $f(z_0) = 0$. Show that there is $r > 0$ such that $\{z : |z - z_0| \leq r\} \subset S$ and $f(z) \neq 0$ for all z with $|z - z_0| = r$.

(c) For $r > 0$ as above, let C denote the circle $\{z : |z - z_0| = r\}$ and set $\varepsilon := 1/2 \cdot \min\{|f(z)| : z \in C\}$. Prove that $f(D(z_0, r))$ contains the open disc $D(0, \epsilon)$: Given $w \in \mathbb{C}$ with $|w| < \varepsilon$, for all $z \in C$ we have

$$|f(z) - w| \geq |f(z)| - |w| \geq \varepsilon\,.$$

On the other hand, $|f(z_0) - w| = |0 - w| = |w| < \varepsilon$. Therefore, the global minimum of the continuous function $|f(\cdot) - w|$ on the closed set $\overline{D(z_0, r)}$ (which exists by the Extreme Value Thm.) must be attained at some point $z' \in D(z_0, r)$. By (a), this minimum must be equal to zero (as f is non-constant), hence $f(z') = w$.

(d) Suppose $f(z_0) \neq 0$. Define $g(z) = f(z) - f(z_0)$ and repeat the argument of (c) for g to show that $f(D(z_0, r))$ contains $D(f(z_0), \varepsilon)$.

(e) Use part (d) to conclude the theorem.

4. Suppose $f \in \mathcal{O}(S)$ is non-constant and $f(S) = T$. Use Problem 3, to prove that if $f(z_0)$ is a boundary point of T then z_0 is a boundary point of S.

5. Suppose f is analytic in some open $\Omega \subset \mathbb{C}$ containing the closed unit disc $\overline{D(0,1)}$ and maps the unit circle $C(0,1)$ into itself. Prove that f maps the open unit disc $D(0,1)$ *onto* itself. [Hint: First, use the Maximum Modulus Principle to show that f maps $D(0,1)$ *into* itself. Then use Problem 4, to show that $f(D(0,1)) \supset D(0,1)$.]

6. Prove that $f(z) = \displaystyle\int_0^1 \frac{\sin(zt)}{t}\,dt$ is an entire function.