Problem Set 7
November 16, 2018.

1. Give a detailed proof of the "Characterization of Analytic Functions" (a.k.a., the "7 Equivalent Conditions") theorem from class.

2. The following is needed to complete the solution of Problem 2 from PS 6:
Let f be an entire function. Prove that f is a polynomial if and only if there exist $R > 0$, $C > 0$ and $N \in \mathbb{Z}_+$ such that $|f(z)| \leq C|z|^N$ for all z with $|z| \geq R$. [Hint: Use the Cauchy Estimates theorem.]

3. For each of the following functions, find its Laurent expansion convergent in some punctured disc centered at z_0, $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$:

(a) $f(z) = \dfrac{1}{z^4 + z^2}$, $z_0 = 0$

(b) $f(z) = \dfrac{e^{1/z^2}}{z}$, $z_0 = 0$

(c) $f(z) = \dfrac{1}{z^2 - 4}$, $z_0 = 2$.

4. For each of the functions in Problem 3, find the integrals

$$\int_{C(z_0,\epsilon)} f(\zeta)d\zeta \quad \text{and} \quad \int_{C(z_0,\epsilon)} f(\zeta)(\zeta - z_0)^{2018} d\zeta\,,$$

where ϵ is some small positive constant. [Hint: Do not attempt a direct computation.]

5. Find *all* different Laurent expansions about $z_0 = 1$ of the function

$$f(z) = \frac{1}{z} - \frac{2z}{z-2} + \frac{z^3}{z+3}\,,$$

and determine their respective annuli of convergence.

6. Given a Laurent series $\sum_{n=-\infty}^{-1} c_n(z - z_0)^n$, set $r = \limsup_{n\to\infty} \sqrt[n]{|c_{-n}|}$. Prove that the function $f(z) = \sum_{n=-\infty}^{-1} c_n(z - z_0)^n$ is well-defined and analytic in $\Omega := \{z \in \mathbb{C} : |z| > r\}$.