Problem Set 8
November 23, 2018.

1. (*Casorati–Weierstrass Theorem*) Suppose $z_0 \in \mathbb{C}$ is an essential singularity of f and $R > 0$ is such that $f \in \mathcal{O}(D(z_0, R) \setminus \{z_0\})$. Prove that, for every $0 < r < R$, we have $\overline{f(D(z_0, r) \setminus \{z_0\})} = \mathbb{C}$ (that is, the set $\{f(z) : z \in D(z_0, r) \setminus \{z_0\}\}$ is dense in $\mathbb{C}$).

[Hint: Given r as above, suppose, for a proof by contradiction, that there are $w \in \mathbb{C}$ and $\delta > 0$ such that $|f(z)-w| > \delta$ for all $z \in D(z_0, r)\setminus\{z_0\}$. Show that then the function $h(z) := \dfrac{1}{f(z) - w}$ is analytic in $D(z_0, r)$. Conclude that then $\lim_{z\to z_0} f(z)$ exists (in $\widehat{\mathbb{C}}$) and so f has a removable singularity or a pole at z_0.]

2. Prove that the image of $\mathbb{C}$ by a non-constant entire function is dense in $\mathbb{C}$.

[Hint: Consider separately the cases when f is polynomial and transcendental (non-polynomial).]

3. Let $P, Q \in \mathbb{C}[z]$ be polynomials with $Q^{-1}(0) \neq \mathbb{C}$ and $\deg(Q) \geq \deg(P) \geq 0$, and let $f(z) = \dfrac{P(z)}{Q(z)}$ for $z \in \mathbb{C} \setminus Q^{-1}(0)$. Prove that $\operatorname{Res}(f, \infty) = 0$.

4. Use Residue Theorem to evaluate the following integrals. Show your work and justify.

(a) $\displaystyle\int_{C(0,2)} ze^{3/z} dz$

(b) $\displaystyle\int_{C(i,2)} \frac{z^3 - 2z}{(z^2 + 4)(10z^{10} - 1)}\, dz$

(c) $\displaystyle\int_{C(0,2018)} \frac{z^{2018} + 2z^{1009} + 1009z^2}{z^{2018} - 2018}\, dz\,.$

5. Suppose $\Omega \subset \mathbb{C}$ is open and contains the upper half-plane $\{z : \operatorname{Im}(z) \geq 0\}$. Let $A = \{a_1, \ldots, a_k\}$ be a finite subset of $\{z : \operatorname{Im}(z) > 0\}$, and let $f \in \mathcal{O}(\Omega \setminus A)$. Suppose further that there are $M > 0$, $R > 0$ and $\alpha > 1$ such that $|f(z)| < \dfrac{M}{|z|^\alpha}$ whenever $|z| > R$. Prove that then the integral $\displaystyle\int_{-\infty}^{+\infty} f(x)dx \; \left(= \lim_{r\to+\infty} \int_{-r}^{r} f(x)dx\right)$ is convergent and satisfies the formula

$$\int_{-\infty}^{+\infty} f(x)dx = 2\pi i \cdot \sum_{j=1}^{k} \operatorname{Res}(f, a_j)\,.$$

[Hint: For $r > \max\{R, |a_j| : j = 1, \ldots, k\}$, consider $\int_{\gamma_r} f(z)dz$ where γ_r is the contour of the upper semi-disc $\{z : |z| \leq r, \operatorname{Im}(z) \geq 0\}$ traversed counter-clockwise, and let r tend to $+\infty$.]

6. Use the above to show that if $P, Q \in \mathbb{R}[x]$ satisfy $Q^{-1}(0) \cap \mathbb{R} = \varnothing$ and $\deg(Q) \geq \deg(P) + 2$ then

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\, dx = 2\pi i \cdot \sum_{j=1}^{k} \operatorname{Res}(\frac{P}{Q}, a_j)\,,$$

where the sum is over all poles of $\frac{P(z)}{Q(z)}$ in the open upper half-plane $\{z : \operatorname{Im}(z) > 0\}$.

7. Evaluate $\displaystyle\int_{0}^{+\infty} \frac{dx}{(1 + x^2)^2}$. Show your work and justify.