Problem Set 9
November 30, 2018.

1. Suppose f is bounded and analytic in $\{z : \operatorname{Im} z > 0\}$, extends continuously to the real axis, and (the extension) is real on the real axis. Prove that f is constant.

2. Suppose f is an entire function, which is real-valued on the real axis and imaginary-valued on the imaginary axis. Prove that f is an odd function (i.e., $f(-z) = -f(z)$ for all $z \in \mathbb{C}$).

3. Show that if f is analytic and non-constant on a bounded domain D, then $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ assume their maxima and minima on the boundary ∂D. [Hint: Use the Open Mapping Theorem.]

4. (a) Prove that the function $f(z) = \dfrac{z-i}{z+i}$ is an analytic bijection between the upper half plane $\{z : \operatorname{Im}(z) > 0\}$ and the open unit disc, with an analytic inverse $g(w) = -i \cdot \dfrac{w+1}{w-1}$.

(b) Prove that there is no non-constant analytic function in the open unit disc (with a continuous extension to the unit circle), which is real-valued on the unit circle.

(c) Prove that there is no function f analytic in the disc $\{z : |z| < 2018\}$ and such that $|f(z)| \to +\infty$ as $|z| \to 2018^-$.

5. Derive the Fundamental Theorem of Algebra as a corollary of Rouché's Theorem.

6. Determine the number of zeros (counted with multiplicities) of the polynomial

$$P(z) = 2z^5 - 6z^2 + z + 1$$

in the annulus $\{z : 1 \leq |z| \leq 2\}$. Justify your answer.

7. Define a function f analytic in $\mathbb{C} \setminus \{x + iy : x \leq 0, y = 0\}$ and such that $f(x) = x^x$ for all $x > 0$. Find $f(i)$ and $f(-i)$. Show that $f(\bar{z}) = \overline{f(z)}$, for all z in the domain of f.