Practice Midterm I

1. Problems from Problem Sets 1 and 2.

2. State definitions of the argument and the principal argument of a non-zero complex number.

3. Find i^i.

4. State the definition of a harmonic function and its harmonic conjugate.

5. State the definition of an analytic function.

6. For each of the following functions $v(x, y)$, find its harmonic conjugate in the right half-plane (if exists) and find a closed form of a function $f(z)$ whose *imaginary* part equals v:

(a) $v(x, y) = \arctan\frac{y}{x}$

(b) $v(x, y) = 6x^2y - 2y^3$

(c) $v(x, y) = e^x \cos(iy)$.

7. Show that if $f = u + iv$ is $\mathbb{C}$-differentiable in an open set Ω and u is constant on Ω, then so is f.

8. Find the set of those $z \in \mathbb{C} \setminus \{0\}$ for which $\mathrm{Arg}(\bar{z}) = \mathrm{Arg}(z^{-1})$. Justify your answer.

9. Use the Local Isolation of Zeros Theorem to prove that the complex exponential function exp (respectively, sin and cos) is the unique extension to $\mathbb{C}$ of the real exponential function exp (respectively, sin and cos) on $\mathbb{R}$.

10. Let Ω be a non-empty domain in $\mathbb{C}$. Prove that if analytic functions $f, g \in \mathcal{O}(\Omega)$ satisfy $f(z)\cdot g(z) = 0$ for all $z \in \Omega$, then at least one of the functions f and g is identically zero on Ω.

11. Let $\Omega \subset \mathbb{C}$ be an open set, let $z_0 \in \Omega$, and let $f : \Omega \to \mathbb{C}$. Prove that f is $\mathbb{C}$-differentiable at z_0 if and only if there exist $r > 0$ and a $\mathbb{C}$-linear function $\varphi : \mathbb{C} \to \mathbb{C}$ such that

$$f(z_0 + h) = f(z_0) + \varphi(h) + o(h),$$

for all $h \in D(0, r)$.