Problem Set 2
January 21, 2024.

1. Exercises 3.9, 3.10.

2. Exercise 4.3.

3. Exercise 4.15.

4. Let X be a set. Let Δ be the collection of all outer measures on X, and let Λ be the collection of all pairs $(\mathcal{M}, \mu)$ such that $\mathcal{M}$ is a σ-algebra on X and μ is a measure on $\mathcal{M}$. For any $\alpha \in \Delta$, let $(\mathcal{M}_\alpha, \alpha_c) \in \Lambda$ denote the pair consisting of α-measurable sets $\mathcal{M}_\alpha$ and the measure $\alpha_c := \alpha|_{\mathcal{M}_\alpha}$. For $(\mathcal{M}, \mu) \in \Lambda$, let $\mu^0 \in \Delta$ denote the effect of Caratheodory construction on μ. Prove the following:

(a) $(\alpha_c)^0 = \alpha$ iff α is regular.

(b) $(\mu^0)_c = \mu$ iff there exists a regular $\gamma \in \Delta$ such that $\mu = \gamma_c$.

(c) If μ is complete and σ-finite, then $(\mu^0)_c = \mu$.

(d) For every $\mu \in \Lambda$, we have $((\mu^0)_c)^0 = \mu^0$.

Remark: In the above problems, equality of measures is understood in the sense of functions; i.e., together with their σ-algebraic domains.