Problem Set 4
February 4, 2024.

1. Let C be a subset of the $[0,1]$ interval defined as $C = \bigcap_{n=1}^{\infty} C_n$, where $C_1 = [0, \frac{1}{5}] \cup [\frac{2}{5}, \frac{3}{5}] \cup [\frac{4}{5}, 1]$ and, for any $k \geq 1$, C_{k+1} is obtained from C_k by removing the second and fourth open fifths from each of the 3^k congruent closed intervals that C_k is composed of.
Let m denote the Lebesgue measure in $\mathbb{R}$. Find $m(C)$ and the Hausdorff dimension of C. Justify your answers.

2. Let C be a subset of the $[0,1]$ interval defined as $C = \bigcap_{n=1}^{\infty} C_n$, where $C_1 = [0, \frac{2}{5}] \cup [\frac{3}{5}, 1]$ and, for any $k \geq 1$, C_{k+1} is obtained from C_k by removing the middle open fifth from each of the 2^k congruent closed intervals that C_k is composed of.
Let m denote the Lebesgue measure in $\mathbb{R}$. Find $m(C)$ and the Hausdorff dimension of C. Justify your answers.

3. Let $S \subset [0,1]^2 \subset \mathbb{R}^2$ denote the *Sierpiński carpet*; i.e., $S = \bigcap_{n=1}^{\infty} S_n$, where

$$S_1 = [0,1]^2 \setminus \left(\frac{1}{3}, \frac{2}{3}\right)^2$$

and, for any $k \geq 1$, S_{k+1} is obtained from S_k by removing the open middle ninth square from each of the 8^k congruent squares of area $\frac{1}{9^k}$ that S_k is composed of.
Let m denote the Lebesgue measure in $\mathbb{R}^2$. Find $m(S)$ and the Hausdorff dimension of S. Justify your answers.

4. Prove that, for every $n \in \mathbb{Z}_+$, the Hausdorff dimension of $\mathbb{R}^n$ equals n.

5. For functions f and g on a measure space $(X, \mathcal{M}, \mu)$, we say that $f = g$ *almost everywhere* (*a.e.*, for short), when $\mu(\{x \in X : f(x) \neq g(x)\}) = 0$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $g : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function.

(a) Suppose that $f = g$ a.e. (with respect to Lebesgue measure). Prove that f is Lebesgue measurable.

(b) Suppose now that f is Lebesgue measurable. Prove that there exists a Borel measurable function $h : \mathbb{R} \to \mathbb{R}$ such that $f = h$ a.e. (with respect to Lebesgue measure).

6. Exercises 5.1–5.3.

NB. For Problems 5 and 6 above, notice that for *real*-valued functions on a measurable space, the notions of measurability as defined in class and in the textbook coincide.