Problem Set 7
March 10, 2024.

For Problems 1 and 2, let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let $f : X \to \mathbb{R}$ be an $\mathcal{M}$-measurable function. Define the *distribution function* of f by

$$\mu_f(t) := \mu(\{x \in X : |f(x)| \geq t\}), \quad t > 0.$$

1. Show that $\mu_f : (0, \infty) \to [0, \mu(X)]$ is non-increasing and Borel measurable.

2. Prove that, for any $p \in [1, \infty)$,

$$\int_X |f(x)|^p d\mu(x) = \int_0^\infty \mu_f(t) p t^{p-1} dt\,.$$

Hint: $|f(x)|^p = \int_0^{|f(x)|} p t^{p-1} dt$.

3. Exercises 11.12–11.16.

4. Exercises 11.18–11.20.