Practice Term Test 1

1. State the definitions of an algebra, σ-algebra, and monotone class of subsets of a set X.

2. For each of the following statements, prove or give a specific counterexample (with justification):

(a) Every algebra $\mathcal{A}$ on a set X is a σ-algebra.
(b) Every σ-algebra $\mathcal{A}$ on a set X is a monotone class.
(c) Every monotone class $\mathcal{M}$ on a set X is a σ-algebra.
(d) If $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots$ are σ-algebras on X, then $\bigcup_i \mathcal{A}_i$ is a σ-algebra on X.
(e) If $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots$ are monotone classes on X, then $\bigcup_i \mathcal{M}_i$ is a monotone class on X.

3. State the definitions of a measurable space, measure, and outer measure.

4. Give an example of a finite set X and an outer measure μ^* on X which is not a measure.

5. Let μ^* be an outer measure on a set X. State the definition of a μ^*-measurable set.

6. Let $X = \mathbb{R}$ and let μ^* be a function on $\mathcal{P}(\mathbb{R})$ which assigns 0 to every countable set, 1 to every set with countable complement, and 1/2 to every other set.

(a) Prove that μ^* is an outer measure.
(b) Identify all the μ^*-measurable sets. Justify your answer.
(c) Prove that μ^* is not a measure on $\mathbb{R}$.

7. (a) State the definition of a regular outer measure.
(b) Let $\mathcal{C}$ be a collection of subsets of a set X such that $\varnothing \in \mathcal{C}$, and let $\zeta : \mathcal{C} \to [0, \infty]$ be a function such that $\zeta(\varnothing) = 0$. State the Caratheodory construction of an outer measure μ^* from ζ.

8. (a) Let $(X, \mathcal{M}, \mu)$ be a measure space. State the definition of a μ-null set.
(b) Let $n \in \mathbb{Z}_+$ and let m (resp. m^*) denote the Lebesgue measure (resp. outer measure) on $\mathbb{R}^n$. Prove that every set $A \subset \mathbb{R}^n$ with $m^*(A) = 0$ is Lebesgue measurable and satisfies $m(A) = 0$.

9. (a) State the definition of Borel measurable sets in a metric space (X, d).
(b) Let $\mathcal{B}$ denote the σ-algebra of Borel measurable sets in $\mathbb{R}$ and let m^* be the Lebesgue outer measure in $\mathbb{R}$. Prove or give a counterexample (with justification): If $A \subset \mathbb{R}$ satisfies $m^*(A) = 0$, then there exist $B, C \in \mathcal{B}$ such that $A = B \setminus C$.

10. (a) Give an example (with justification) of a set X and a finite outer measure μ^* on X, subsets $A_n \uparrow A$ of X, and subsets $B_n \downarrow B$ of X such that $\mu^*(A_n)$ does not converge to $\mu^*(A)$ and $\mu^*(B_n)$ does not converge to $\mu^*(B)$.
(b) Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and let μ^* be the Caratheodory extension of μ. Show that if $A_n \uparrow A$ for subsets A_n, A of X, then $\mu^*(A) = \lim_{n\to\infty} \mu^*(A_n)$.

11. Let $(X, \mathcal{M})$ be a measurable space and let $f : X \to \overline{\mathbb{R}}$.

(a) Prove that f is a measurable function if and only if $f^{-1}(B) \in \mathcal{M}$ for every Borel measurable $B \subset \overline{\mathbb{R}}$.
(b) Prove that f is a simple function if and only if $f(X)$ is a finite set, $f(X) \subset \mathbb{R}$ and f is measurable.

12. Let m^* denote the Lebesgue outer measure in $\mathbb{R}^2$. Find $m^*(A)$ for the sets A from the following list. Which of the sets are m^*-measurable? Justify your answers.

(a) $A = \mathbb{Q} \times \mathbb{R}$.
(b) $A = \mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})$.
(c) $A = \mathbb{Q} \times V$, where V is a Vitali set in $[0, 1]$.
(d) $A = (\mathbb{R} \setminus \mathbb{Q}) \times V$, where V is a Vitali set in $[0, 1]$.

13. Prove that the Lebesgue integral of a non-negative simple function is well defined; i.e., independent of its representation as a non-negative combination of characteristic functions (Exercise 6.1).