MEASURE THEORY (MATH 4122/9022)

I. ALGEBRAS, σ-ALGEBRAS & MONOTONE CLASSES

X = arbitrary set. $\mathcal{P}(X)$ = "power set" of X = set of all subsets of X.

Def. An algebra (of subsets of X) is a set $\mathcal{A} \subset \mathcal{P}(X)$ such that
(i) $\emptyset \in \mathcal{A}$
(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
(iii) $A_1, \dots, A_k \in \mathcal{A} \Rightarrow A_1 \cup \dots \cup A_k \in \mathcal{A}$.

Remark: If $\mathcal{A}$ is an algebra, then (i') $X \in \mathcal{A}$, and (iii') $A_1, \dots, A_k \in \mathcal{A} \Rightarrow A_1 \cap \dots \cap A_k \in \mathcal{A}$.
(Hence also: $A, B \in \mathcal{A} \Rightarrow A \setminus B \in \mathcal{A}$.)

Def. A σ-algebra (of subsets of X) is a set $\mathcal{A} \subset \mathcal{P}(X)$ such that $\mathcal{A}$ is an algebra, and (iv) $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Remark: $\mathcal{A}$ a σ-algebra $\Rightarrow \left[\{A_n\}_{n=1}^{\infty} \subset \mathcal{A} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{A} \right]$.

Def. If $\mathcal{A}$ is a σ-algebra on X, then the pair $(X, \mathcal{A})$ is called a measurable space. Elements of $\mathcal{A}$ are called measurable sets.

Examples. 1) X a set. $\mathcal{P}(X)$ is a σ-algebra on X (the largest).
2) X a set. $\mathcal{A} := \{\emptyset, X\}$ is the smallest σ-algebra on X.
3) X a set. $\mathcal{A} := \{A \subset X : |A| \le \aleph_0 \text{ or } |A^c| \le \aleph_0\}$ is a σ-algebra on X.
4) X a finite set, $\mathcal{A}$ a σ-algebra (equiv.: algebra) on X.
Consider an equiv. relation on X: $x \sim y \Leftrightarrow [\forall A \in \mathcal{A} : x \in A \Leftrightarrow y \in A]$.
Then, $\forall x \in X$, $[x]_\sim = \bigcap \{A \in \mathcal{A} : x \in A\}$, and so $[x]_\sim \in \mathcal{A}$ (b/c X is finite).
Let $A_1, \dots, A_p$ be all the equiv. classes mod. $\sim$. Then $\mathcal{A} = \{$finite unions of subsets of $\{A_1, \dots, A_p\}\}$
(Exercise)

Observation: If $\{\mathcal{A}_\alpha\}$ is a collection of σ-algebras on X, then $\bigcap_\alpha \mathcal{A}_\alpha$ is a σ-alg. on X.

Def. Given a collection of sets $\mathcal{C} \subset \mathcal{P}(X)$, the σ-algebra
$\sigma(\mathcal{C}) := \bigcap \{\mathcal{A} : \mathcal{A} \text{ a } \sigma\text{-algebra}, \mathcal{C} \subset \mathcal{A}\}$ is called the σ-algebra generated by $\mathcal{C}$.

Def. Given a topological space (X,τ), the σ-algebra $\mathcal{B}(X) := \sigma(\tau)$ generated by the open sets of X is called the <u>Borel σ-algebra</u> on X.
The elements of $\mathcal{B}(X)$ are called <u>Borel measurable</u>.

Def. (X,τ) a topological space. A set $A \subset X$ is called:

- a <u>G_δ set</u>, if $\exists \{G_n\}_{n=1}^\infty \subset \tau$ s.t. $A = \bigcap_{n=1}^\infty G_n$
- an <u>F_σ set</u>, if $\exists \{F_n\}_{n=1}^\infty$ closed s.t. $A = \bigcup_{n=1}^\infty F_n$
- a <u>σ-compact</u> set, if $\exists \{K_n\}_{n=1}^\infty$ compact s.t. $A = \bigcup_{n=1}^\infty K_n$.

<u>Remark</u>. The G_δ and F_σ sets are Borel measurable in any top. space. In a Hausdorff space, so are σ-compacts.

<u>**Prop.**</u> If $X = \mathbb{R}$, then $\mathcal{B}(X)$ is generated by any of the following

(a) $C_1 = \{(a,b) : a,b \in \mathbb{R}\}$
(b) $C_2 = \{[a,b] : a,b \in \mathbb{R}\}$
(c) $C_3 = \{(a,b] : a,b \in \mathbb{R}\}$
(d) $C_4 = \{(a,\infty) : a \in \mathbb{R}\}$.

Pf. Let τ denote the Euclidean top. on X. Then $C_1 \subset \tau$, so $\sigma(C_1) \subset \sigma(\tau) = \mathcal{B}(X)$.
On the other hand, $\forall U \in \tau$, $\exists \{(a_n,b_n)\}_{n=1}^\infty$ s.t. $U = \bigcup_{n=1}^\infty (a_n,b_n)$, and so $U \in \sigma(C_1)$.
Thus, $\tau \subset \sigma(C_1)$, and hence $\mathcal{B}(X) = \sigma(\tau) \subset \sigma(\sigma(C_1)) = \sigma(C_1)$. So, $\sigma(C_1) = \mathcal{B}(X)$.
Next, for any $a,b \in \mathbb{R}$, $[a,b] = \bigcap_{n=1}^\infty (a-\frac{1}{n}, b+\frac{1}{n}) \in \sigma(C_1)$, so $C_2 \subset \sigma(C_1)$, and hence $\sigma(C_2) \subset \sigma(\sigma(C_1)) = \sigma(C_1)$. Conversely, $(a,b) = \bigcup_{n \geq n_0} [a+\frac{1}{n}, b-\frac{1}{n}]$, where $n_0 \geq \frac{b-a}{2}$, so $C_1 \subset \sigma(C_2)$ and hence $\sigma(C_1) \subset \sigma(\sigma(C_2)) = \sigma(C_2)$.
The other inclusions are analogous. $\square$

Def. A <u>monotone class</u> (on X) is a set $\mathcal{M} \subset \mathcal{P}(X)$ such that

(i) $\{A_n\}_1^\infty \subset \mathcal{M}$, $A_n \uparrow A \Rightarrow A \in \mathcal{M}$
(ii) $\{A_n\}_1^\infty \subset \mathcal{M}$, $A_n \downarrow A \Rightarrow A \in \mathcal{M}$.

<u>Notation</u>: $A_n \uparrow A :\Leftrightarrow A_k \subset A_{k+1}, \forall k$ and $A := \bigcup_{k=1}^\infty A_k$.
$A_n \downarrow A :\Leftrightarrow A_k \supset A_{k+1}, \forall k$ and $A := \bigcap_{k=1}^\infty A_k$

<u>Obs.</u> If $\{\mathcal{M}_\alpha\}$ are monotone classes in X,
then $\bigcap_\alpha \mathcal{M}_\alpha$ is a monotone class in X.

Def. Given a collection of sets $\mathcal{C} \subset \mathcal{P}(X)$, the smallest monotone class $\mathcal{M}$ s.t. $\mathcal{C} \subset \mathcal{M}$ is called the <u>monotone class generated by $\mathcal{C}$</u>.

<u>Remarks</u>: 1) Every σ-algebra is a monotone class, by def'n.

2) If a monotone class $\mathcal{M}$ is an algebra, then $\mathcal{M}$ is a σ-algebra.

3) Example: The family $\{I \subset \mathbb{R} : I \text{ is an interval}\}$ is a monotone class but <u>not</u> a σ-algebra.

<u>Thm.</u> (<u>Monotone Class Thm.</u>) Let X be a set and let $\mathcal{A}$ be an algebra on X. If $\mathcal{M}$ is the monotone class generated by $\mathcal{A}$, then $\mathcal{M} = \sigma(\mathcal{A})$.

Pf. By Rem. 1) above, $\sigma(\mathcal{A})$ is a monotone class containing $\mathcal{A}$, so $\mathcal{M} \subset \sigma(\mathcal{A})$.
It thus suffices to show that $\mathcal{M}$ is a σ-algebra (b/c then $\sigma(\mathcal{A}) \subset \mathcal{M}$, by minimality), and thus, by Rem. 2), it suffices to show that $\mathcal{M}$ is an algebra.

For any $\mathcal{L} \subset \mathcal{P}(X)$, define $\mathcal{J}(\mathcal{L}) = \{A \subset X \mid A \cup B, A \setminus B, B \setminus A \in \mathcal{M} \text{ for all } B \in \mathcal{L}\}$.

Then, we have: (1) $\mathcal{L} \subset \mathcal{P}(X) \Rightarrow \mathcal{J}(\mathcal{L})$ is a monotone class.

(2) $\mathcal{L}, \mathcal{K} \subset \mathcal{P}(X) \Rightarrow \left[\mathcal{L} \subset \mathcal{J}(\mathcal{K}) \Leftrightarrow \mathcal{K} \subset \mathcal{J}(\mathcal{L})\right]$.

Indeed, 1) follows from monotonicity of $\mathcal{M}$ (and de Morgan laws), and 2) follows as

$$\mathcal{L} \subset \mathcal{J}(\mathcal{K}) \Leftrightarrow \forall A \in \mathcal{L}\,(\forall B \in \mathcal{K}:\ A \cup B, A \setminus B, B \setminus A \in \mathcal{M}) \Leftrightarrow \forall B \in \mathcal{K}\,(\forall A \in \mathcal{L}:\ B \cup A, B \setminus A, A \setminus B \in \mathcal{M}) \Leftrightarrow \mathcal{K} \subset \mathcal{J}(\mathcal{L}).$$

Now, by def'n of $\mathcal{J}(\cdot)$ and b/c $\mathcal{A}$ is an algebra, we get $\mathcal{A} \subset \mathcal{J}(\mathcal{A})$.
Then, by (1) and minimality of $\mathcal{M}$, we have $\mathcal{M} \subset \mathcal{J}(\mathcal{A})$. Hence, $\mathcal{A} \subset \mathcal{J}(\mathcal{M})$, by (2), and thus $\mathcal{M} \subset \mathcal{J}(\mathcal{M})$, again by (1) and minimality of $\mathcal{M}$.

It follows that $X \in \mathcal{M}$ (b/c $\mathcal{A} \subset \mathcal{M}$) and $A \cup B, A \setminus B \in \mathcal{M}$ for all $A, B \in \mathcal{M}$, by def'n of $\mathcal{J}(\cdot)$. Therefore $\mathcal{M}$ is indeed an algebra. $\square$

II. MEASURES.

Def. Let $(X, \mathcal{M})$ be a measurable space. A function $\mu : \mathcal{M} \to [0, \infty]$ is called a (positive) <u>measure</u> (on $(X, \mathcal{M})$, or simply on X) when

(i) $\mu(\emptyset) = 0$

(ii) $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$, $A_k \cap A_j = \emptyset \ \forall k \neq j \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ /countable additivity/

If μ is a measure on $(X, \mathcal{M})$ then the triple $(X, \mathcal{M}, \mu)$ is called a <u>measure space</u>.

Def. Let $(X, \mathcal{M}, \mu)$ be a measure space. We say that a set $A \subset X$ is <u>of σ-finite measure</u> if $\exists \{A_n\} \subset \mathcal{M}$ s.t. $A = \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty$, $\forall n$.

The measure μ is called

- <u>finite</u>, when $\mu(X) < \infty$
- <u>σ-finite</u>, when $\exists \{A_n\} \subset \mathcal{M}$ s.t. $X = \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty$, $\forall n$.

Examples: 1) X a set, $\mathcal{M} = \{\emptyset, X\}$, $\mu(\emptyset) = 0$, $\mu(X) = a$ for some $a \in [0, \infty]$.

2) X a set, $\mathcal{M} = \mathcal{P}(X)$, $\mu : \mathcal{P}(X) \ni A \mapsto \mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite.} \end{cases}$

This μ is called a <u>counting measure</u>.

3) $X = \mathbb{N}$, $(a_n)_{n=0}^{\infty}$ sequence of real numbers s.t. $a_n \geq 0$, $\forall n$, $\sum_{n=0}^{\infty} a_n < \infty$; $\mathcal{M} = \mathcal{P}(X)$.

$\mu : \mathcal{P}(X) \ni A \mapsto \mu(A) = \sum_{n \in A} a_n$. Then, μ is a finite measure on $\mathbb{N}$.

4) $X = \mathbb{R}$, $\mathcal{M} = \mathcal{P}(X)$, $x_0 \in X$ fixed. $\mu : \mathcal{P}(X) \ni A \mapsto \mu(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{if } x_0 \notin A \end{cases}$

Notation: $\mu = \delta_{x_0}$ is called the <u>point mass</u> at x_0.

5) $X = \mathbb{R}$, $\mathcal{M} = \{A \subset X : |A| \leq \aleph_0 \text{ or } |A^c| \leq \aleph_0\}$, $\mu : \mathcal{M} \ni A \mapsto \mu(A) = \begin{cases} 0 & \text{if } |A| \leq \aleph_0 \\ 1 & \text{if } |A^c| \leq \aleph_0 \end{cases}$

Prop. (<u>Basic Properties of Measures</u>) Let $(X, \mathcal{M}, \mu)$ be a measure space. Then:

(i) $A_1, \dots, A_k \in \mathcal{M}$, $A_i \cap A_j = \emptyset$ for all $i \neq j \Rightarrow \mu\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \mu(A_i)$ /finite additivity/

(ii) $A, B \in \mathcal{M}$, $A \subset B \Rightarrow \mu(A) \leq \mu(B)$ /monotonicity/

(iii) $A, B \in \mathcal{M}$, $A \subset B$, $\mu(B) < \infty \Rightarrow \mu(B \setminus A) = \mu(B) \setminus \mu(A)$.

(iv) $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M} \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$ /subadditivity/

(v) $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$, $A_n \uparrow A \Rightarrow \mu(A) = \lim_{n \to \infty} \mu(A_n)$

(vi) $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$, $A_n \downarrow A$, $\exists n_0$ s.t. $\mu(A_{n_0}) < \infty \Rightarrow \mu(A) = \lim_{n \to \infty} \mu(A_n)$.

Pf. (i) Given $A_1,\dots,A_k$ pairwise disjoint, set $A_{k+1}=A_{k+2}=\dots=\emptyset$. Then $\mu(A_i)=0$, $\forall i>k$, and by countable additivity of μ, $\mu(A_1\cup\dots\cup A_k)=\mu(\bigcup_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}\mu(A_n)=\mu(A_1)+\dots+\mu(A_k)$. ✓

(ii) If $A\subset B$, then $B=A\amalg(B\setminus A)$, so by (i), $\mu(A)+\mu(B\setminus A)=\mu(B)$, hence $\mu(B)\geq\mu(A)$. ✓

(iii) By above, if $\mu(B)<\infty$ then $\mu(A),\mu(B\setminus A)<\infty$, and $\mu(B\setminus A)=\mu(B)-\mu(A)$. ✓

(iv) Given $\{A_n\}\subset\mathcal{M}$, set $B_1:=A_1$, $B_2:=A_2\setminus A_1,\dots$, $B_i:=A_i\setminus\bigcup_{j=1}^{i-1}A_j,\dots$

Then $\{B_n\}\subset\mathcal{M}$ and $B_i\cap B_j=\emptyset$ for $i\neq j$, so $\mu(\bigcup_{n=1}^{\infty}A_n)=\mu(\bigcup_{n=1}^{\infty}B_n)=\sum_{n=1}^{\infty}\mu(B_n)\leq\sum_{n=1}^{\infty}\mu(A_n)$

by (ii), as $B_i\subset A_i$.

(v) Given $\{A_n\}\subset\mathcal{M}$, $A_n\uparrow A$, set $\{B_n\}$ as above. Then, $\forall k$,

$B_k\subset A_k$ and $B_i\cap B_k=\emptyset$ for all $i<k$ (b/c $B_i\subset A_i\subset A_k$ and $B_k=A_k\setminus(A_1\cup\dots\cup A_{k-1})$).

Thus, A_k is a disjoint union of $B_1,\dots,B_k$, and so $\mu(A_k)=\sum_{i=1}^{k}\mu(B_i)\xrightarrow[k\to\infty]{}\sum_{i=1}^{\infty}\mu(B_i)=\mu(\bigcup_{i=1}^{\infty}B_i)$ pairwise disjoint $=\mu(\bigcup_{i=1}^{\infty}A_i)$. ✓

(vi) Given $\{A_n\}\subset\mathcal{M}$, $A_n\downarrow A$, let n_0 be s.t. $\mu(A_{n_0})<\infty$.

Then, $\mu(\bigcap_{n=1}^{\infty}A_n)=\mu(\bigcap_{n\geq n_0}A_n)\leq\mu(A_{n_0})<\infty$, by monotonicity.

The sequence $(A_{n_0}\setminus A_n)_{n\geq n_0}$ is increasing, and hence

$$\underbrace{\mu(A_{n_0})-\mu(A_n)=\mu(A_{n_0}\setminus A_n)}_{\text{by (iii)}}\xrightarrow[n\to\infty]{\text{by (v)}}\mu\Big(\bigcup_{n\geq n_0}(A_{n_0}\setminus A_n)\Big)=\underbrace{\mu\Big(A_{n_0}\setminus\bigcap_{n\geq n_0}A_n\Big)=\mu(A_{n_0})-\mu\Big(\bigcap_{n=1}^{\infty}A_n\Big)}_{\text{by (iii)}},$$

so that $\mu(A_n)\xrightarrow[n\to\infty]{}\mu(\bigcap_{n=1}^{\infty}A_n)$, again by finiteness of $\mu(A_{n_0})$. □

Example: The assumption $\mu(A_{n_0})<\infty$ in (vi) is necessary. Consider, e.g., the counting measure on $X=\mathbb{N}$, and $A_n:=\mathbb{N}\setminus\{0,\dots,n\}$, $n\geq1$. Then, $\bigcap_{n\geq1}A_n=\emptyset$, so it has measure zero, but $\forall n$, $\mu(A_n)=\infty$.

Def. Let $(X,\mathcal{M},\mu)$ be a measure space. A set $B\subset X$ is called a **null set**, if there exists $A\in\mathcal{M}$ s.t. $B\subset A$ and $\mu(A)=0$. If $\mathcal{M}$ contains all null sets (w.r.t. μ), then we say that the measure space $(X,\mathcal{M},\mu)$ is **complete**. The **completion** of $\mathcal{M}$ is the smallest σ-algebra $\overline{\mathcal{M}}$ containing $\mathcal{M}$ s.t. $(X,\overline{\mathcal{M}},\bar\mu)$ is a complete measure space, where $\bar\mu$ is an extension of μ (i.e., $\bar\mu(A)=\mu(A)$ for $A\in\mathcal{M}$).

Remark. Completion of measure always exists and unique. See Ex. 3.8 in [Bass].

III. OUTER MEASURES.

Def. An outer measure on a set X is a function $\mu^*: \mathcal{P}(X) \to [0,\infty]$ satisfying:

(i) $\mu^*(\emptyset) = 0$

(ii) $Y \subset X$, $\{Y_n\}_{n=1}^{\infty} \subset \mathcal{P}(X)$, $Y \subset \bigcup_{n=1}^{\infty} Y_n \Rightarrow \mu^*(Y) \leq \sum_{n=1}^{\infty} \mu^*(Y_n)$.

Remark: Condition (ii) above is equivalent to the conjunction of

(iii) $A, B \subset X$, $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$ /monotonicity/

(iv) $\{A_n\}_{n=1}^{\infty} \subset \mathcal{P}(X) \Rightarrow \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$ /subadditivity/

Example: Let $X = \mathbb{R}$ and set $\mu^*(Y) := \begin{cases} 0 & \text{if } |Y| \leq \aleph_0 \\ 1/2 & \text{if } |Y| > \aleph_0 \text{ and } |Y^c| > \aleph_0 \\ 1 & \text{if } |Y^c| \leq \aleph_0. \end{cases}$

Then, μ^* is an outer measure, which is not a measure on $\mathcal{P}(X)$. (Exercise)

(!) Example (Caratheodory Construction): Let X be a set and let $\mathcal{F} \subset \mathcal{P}(X)$ be s.t. $\emptyset \in \mathcal{F}$ [and $X = \bigcup \mathcal{F}$] redundant! Given a function $\zeta: \mathcal{F} \to [0,\infty]$ satisfying $\zeta(\emptyset) = 0$, define $\mu^* \equiv \mu^*_{\mathcal{F},\zeta}: \mathcal{P}(X) \to [0,\infty]$ by $\mu^*(Y) := \inf\left\{\sum_{n=1}^{\infty} \zeta(E_n) : \{E_n\}_{n=1}^{\infty} \subset \mathcal{F} \text{ s.t. } Y \subset \bigcup_{n=1}^{\infty} E_n\right\}$.

Prop. The above defined μ^* is an outer measure on X.

Pf. Axiom (i) of outer measure is satisfied since $\emptyset \in \mathcal{F}$ and $\zeta(\emptyset) = 0$. ✓

To verify (ii), fix $Y \subset X$ and $\{Y_n\}_{n=1}^{\infty} \subset \mathcal{P}(X)$ s.t. $Y \subset \bigcup_{n=1}^{\infty} Y_n$. Let $\varepsilon > 0$ be arbitrary.

For each $n \geq 1$, choose $\{E_{n,k}\}_{k=1}^{\infty} \subset \mathcal{F}$ s.t. $Y_n \subset \bigcup_{k=1}^{\infty} E_{nk}$ and $\sum_{k=1}^{\infty} \zeta(E_{nk}) \leq \mu^*(Y_n) + \frac{\varepsilon}{2^n}$.

Then, $Y \subset \bigcup_n Y_n \subset \bigcup_{n,k} E_{nk}$ and hence (by countability of $\{E_{nk}\}_{n,k=1}^{\infty}$):

$$\mu^*(Y) \leq \sum_{n,k=1}^{\infty} \zeta(E_{nk}) = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \zeta(E_{nk})\right) \leq \sum_{n=1}^{\infty}\left(\mu^*(Y_n) + \frac{\varepsilon}{2^n}\right) = \left(\sum_{n=1}^{\infty} \mu^*(Y_n)\right) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\mu^*(Y) \leq \sum_n \mu^*(Y_n)$, as required. □

Remark: In general, an outer measure resulting from the Caratheodory construction is not a measure on $\mathcal{P}(X)$. To obtain a measure, one needs to shrink the σ-algebra, as follows.

Def. (Caratheodory Condition) Given an outer measure μ^* on a set X, a subset $A \subset X$ is called μ^*-measurable if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for every $E \subset X$.

Remarks: 1) By symmetry, A is μ^*-measurable iff A^c is so.

2) In the Carathéodory Condition, we always have $\mu^*(E) \le \mu^*(E\cap A)+\mu^*(E\cap A^c)$, so it suffices to check the other inequality.

3) It thus suffices to check the Carathéodory Cond on sets E with $\mu^*(E)<\infty$

Thm. Let X be a set and let μ^* be an outer measure on X

If $\mathcal{M}=\{A\subset X : A \text{ is } \mu^*\text{-measurable}\}$, then:

(1) $\mathcal{M}$ is a σ-algebra

(2) $A\subset X$, $\mu^*(A)=0 \Rightarrow A\in\mathcal{M}$ (i.e., $\mathcal{M}$ contains all the null sets of μ^*)

(3) $\mu := \mu^*|_{\mathcal{M}}$ is a complete measure on $\mathcal{M}$.

Pf. We begin with some preparatory remarks.

1°. $A,B\in\mathcal{M} \Rightarrow A\cup B\in\mathcal{M}$. Indeed, for $E\subset X$, we have $\mu^*(E) \overset{\text{C.C. for } A}{=} \mu^*(E\cap A)+\mu^*(E\cap A^c) \overset{\text{C.C. for } B}{=}$

$$= \mu^*(E\cap A)+\mu^*(E\cap A^c\cap B)+\mu^*(E\cap A^c\cap B^c) =$$

$$= \mu^*(E\cap(A\cup B)\cap A)+\mu^*(E\cap(A\cup B)\cap A^c)+\mu^*(E\cap(A\cup B)^c) \overset{\text{C.C. for } A}{=}$$

$$= \mu^*(E\cap(A\cup B))+\mu^*(E\cap(A\cup B)^c). \checkmark$$

2° $A\in\mathcal{M} \Rightarrow A^c\in\mathcal{M}$. This follows from Rem. 1) above ✓

3° $A,B\in\mathcal{M} \Rightarrow A\setminus B\in\mathcal{M}$. Indeed, $A\setminus B = A\cap B^c = (A^c\cup B)^c$, so the claim follows from 1° & 2°. ✓

4° $A\in\mathcal{M}$, $B\subset X$ s.t. $B\cap A=\emptyset$, $E\subset X \Rightarrow \mu^*(E\cap(A\cup B)) = \mu^*(E\cap A)+\mu^*(E\cap B)$.

Indeed, by C.C for A, we have: $\mu^*(E\cap(A\cup B)) = \mu^*(E\cap(A\cup B)\cap A)+\mu^*(E\cap(A\cup B)\cap A^c)$

$$= \mu^*(E\cap A)+\mu^*(E\cap B), \text{ b/c } B\subset A^c. \checkmark$$

5° $\{A_n\}_{n=1}^\infty\subset\mathcal{M}$, $A_i\cap A_j=\emptyset$ $\forall i\neq j$, $E\subset X \Rightarrow \mu^*(E\cap\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu^*(E\cap A_n)$

Indeed, by 4° and induction, we have $\forall n\ge 1$: $\mu^*(E\cap\bigcup_{k=1}^n A_k) = \sum_{k=1}^n \mu^*(E\cap A_k)$.

Thus, by monotonicity, $\mu^*(E\cap\bigcup_{k=1}^\infty A_k) \ge \sum_{k=1}^n \mu^*(E\cap A_k)$, for every $n\ge 1$.

Hence, letting $n\to\infty$, $\mu^*(E\cap\bigcup_{k=1}^\infty A_k) \ge \sum_{k=1}^\infty \mu^*(E\cap A_k)$. The other inequality always holds, by subadditivity. ✓

6° $\{A_n\}_{n=1}^\infty\subset\mathcal{M}$, $A_i\cap A_j=\emptyset$ $\forall i\neq j \Rightarrow \bigcup_{n=1}^\infty A_n\in\mathcal{M}$. Indeed, pick any $E\subset X$.

By 1° and induction, we have $\mu^*(E) = \mu^*(E\cap\bigcup_{k=1}^n A_k)+\mu^*(E\cap(\bigcup_{k=1}^n A_k)^c)$, $\forall n$

Thus, by 4° and monotonicity, $\mu^*(E) \ge \sum_{k=1}^n \mu^*(E\cap A_k)+\mu^*(E\cap(\bigcup_{k=1}^\infty A_k)^c)$, $\forall n$. Letting $n\to\infty$, we get $\mu^*(E) \geq \sum_{k=1}^{\infty} \mu^*(E \cap A_k) + \mu^*(E \cap (\bigcup_{k=1}^{\infty} A_k)^c)$, and hence, by 5°,

$\mu^*(E) \geq \mu^*(\bigcup_{k=1}^{\infty}(E \cap A_k)) + \mu^*(E \cap (\bigcup_{k=1}^{\infty} A_k)^c)$, as required. ✓

Now, for the proof of (1), note that:

(i) $\phi \in \mathcal{M}$, b/c $\forall E \subset X$, $\mu^*(E) = 0 + \mu^*(E) = \mu^*(\phi) + \mu^*(E) = \mu^*(E \cap \phi) + \mu^*(E \cap \phi^c)$. ✓

(ii) $A \in \mathcal{M}$ iff $A^c \in \mathcal{M}$, by 2°.

(iii) Suppose $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$. Set $B_1 := A_1$, $B_2 := A_2 \setminus A_1, \ldots, B_k := A_k \setminus \bigcup_{i=1}^{k-1} A_i$.

Then, $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$ and each B_n is in $\mathcal{M}$, by 1° & 3°. Thus, $\bigcup_n A_n = \bigcup_n B_n \in \mathcal{M}$, by 6°.

For the proof of (2), suppose $A \subset X$ is s.t. $\mu^*(A) = 0$, and pick $E \subset X$.

By monotonicity, $\mu^*(E \cap A) \leq \mu^*(A) = 0$ and $\mu^*(E \cap A^c) \leq \mu^*(E)$. Hence,

$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E)$. ✓

Finally, for (3) it suffices to show that μ is countably additive on $\mathcal{M}$, since the completeness follows from (2):

Given $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$ with $A_i \cap A_j = \phi$ $\forall i \neq j$, pick $E := X$. Then, by 5°,

$\mu(\bigcup_{n=1}^{\infty} A_n) = \mu^*(X \cap \bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu^*(X \cap A_n) = \sum_{n=1}^{\infty} \mu(A_n)$, which completes the proof. □

In the opposite direction, every measure on a σ-algebra $\mathcal{M} \subset \mathcal{P}(X)$ gives rise to an outer measure on $\mathcal{P}(X)$. Even more generally, every measure on an algebra does so.

__Def.__ Let $\mathcal{A}$ be an algebra of subsets of X. A function $\zeta : \mathcal{A} \to [0, \infty]$ is called a __measure on $\mathcal{A}$__ (or a __premeasure__), when: (i) $\zeta(\phi) = 0$,

(ii) if $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ are s.t. $A_i \cap A_j = \phi$ $\forall i \neq j$ and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, then $\zeta(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \zeta(A_n)$.

__Def.__ An outer measure μ^* is called __regular__, when $\forall Y \subset X$ $\exists A \in \mathcal{M}$ s.t. $Y \subset A$ & $\mu^*(Y) = \mu^*(A)$, where $\mathcal{M}$ is the σ-algebra of μ^*-measurable sets.

__Thm.__ (__Caratheodory Extension Thm.__) Let $\mathcal{A}$ be an algebra on a set X and let μ be a premeasure on $\mathcal{A}$. Define $\mu^*(Y) := \inf \{ \sum_{n=1}^{\infty} \mu(A_n) : \{A_n\} \subset \mathcal{A}, Y \subset \bigcup_{n=1}^{\infty} A_n \}$, for $Y \subset X$.

Then:
1) μ^* is a regular outer measure
2) $\mu^*(A) = \mu(A)$, $\forall A \in \mathcal{A}$
3) the μ^*-measurable sets σ-algebra, $\mathcal{M}$ contains $\mathcal{A}$ and all the null sets of μ^*
4) if μ is σ-finite, then it has a unique extension to $\sigma(\mathcal{A})$.

⑤

Pf. By the proposition following the Carathéodory construction, μ^* is an outer measure. Let then $\mathcal{M}$ denote the σ-algebra of μ^*-measurable sets, and let $\sigma(\mathcal{A})$ be the σ-algebra generated by $\mathcal{A}$.

For the proof of 2), pick $A \in \mathcal{A}$.

Then $\{A_n\}$ defined as $A_1 := A$, $A_n := \emptyset$ $\forall n \geq 2$ is a family from $\mathcal{A}$, and so $\mu^*(A) \leq \mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$, by definition. For the other inequality, let $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ be s.t. $A \subset \bigcup_{n=1}^{\infty} A_n$.

Set $B_1 := A \cap A_1$, $B_2 := A \cap (A_2 \setminus A_1)$, ..., $B_k := A \cap (A_k \setminus \bigcup_{i=1}^{k-1} A_i)$, ... Then the $\{B_n\}_{n=1}^{\infty}$ are pairwise disjoint elements of $\mathcal{A}$ with $\bigcup_{n=1}^{\infty} B_n = A \cap \bigcup_{n=1}^{\infty} A_n = A \in \mathcal{A}$, so

$\mu(A) = \mu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$. Taking inf over all such $\{A_n\}$, we get

$\mu(A) \leq \mu^*(A)$. ✓

For the proof of 3), pick $A \in \mathcal{A}$ and let $E \subset X$ be arbitrary. Let $\varepsilon > 0$ be arbitrary, and let $\{B_n\}_{n=1}^{\infty} \subset \mathcal{A}$ be s.t. $E \subset \bigcup_n B_n$ and $\sum_{n=1}^{\infty} \mu(B_n) \leq \mu^*(E) + \varepsilon$. Then,

$$\mu^*(E) + \varepsilon \geq \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} \mu(B_n \cap A) + \mu(B_n \cap A^c) = \sum_{n=1}^{\infty} \mu(B_n \cap A) + \sum_{n=1}^{\infty} \mu(B_n \cap A^c) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. ✓

That $\mathcal{M}$ contains μ^*-null sets is a special case of (2) in the previous theorem. ✓

For the proof of regularity of μ^*, pick $Y \subset X$ with $\mu^*(Y) < \infty$. For every $n \geq 1$, let $\{A_{nk}\}_{k=1}^{\infty} \subset \mathcal{A}$ be s.t. $Y \subset \bigcup_{k=1}^{\infty} A_{nk}$ and $\sum_{k=1}^{\infty} \mu(A_{nk}) \leq \mu^*(Y) + \frac{1}{n}$. Then, $\forall n$, $B_n := \bigcup_{k=1}^{\infty} A_{nk} \in \sigma(\mathcal{A}) \subset \mathcal{M}$, and so $B := \bigcap_{n \geq 1} B_n \in \sigma(\mathcal{A}) \subset \mathcal{M}$. Of course, $Y \subset B$, and since $\mu^*|_{\mathcal{M}}$ is a measure, we have $\mu^*(B) = \lim_{n\to\infty} \mu^*(\bigcap_{k=1}^{n} B_k) \leq \lim_{n\to\infty} \mu^*(B_n) \leq \lim_{n\to\infty} (\mu^*(Y) + \frac{1}{n}) = \mu^*(Y)$. ✓

Finally, for the proof of 4), suppose that ν is another measure on $\sigma(\mathcal{A})$ with $\nu|_{\mathcal{A}} = \mu$. Note that it suffices to assume that $\mu^*(X) = \nu(X) < \infty$. Indeed, for if $X = \bigcup_{n=1}^{\infty} X_n$ with $X_n \in \mathcal{A}$, $X_n \uparrow X$ and $\mu(X_n) < \infty$, then $\mu^*|_{X_n} \equiv \nu|_{X_n}$ $\forall n$ $\Rightarrow$ $\mu^* \equiv \nu$ on X.

Pick $A \in \sigma(\mathcal{A})$. Then $\mu^*(A) = \inf\{\sum_{n=1}^{\infty} \mu(A_n) : \{A_n\} \subset \mathcal{A}, A \subset \bigcup_{n=1}^{\infty} A_n\}$. But $\mu(A_n) = \nu(A_n)$, $\forall n$, so by monotonicity and subadditivity of ν, $\nu(A) \leq \sum_{n=1}^{\infty} \nu(A_n) = \sum_{n=1}^{\infty} \mu(A_n)$. Taking inf over all such families, we get $\nu(A) \leq \mu^*(A)$.

Similarly, $\nu(A^c) \leq \mu^*(A^c)$, and since $\nu(X) = \mu^*(X) < \infty$, we can write

$\nu(A) = \nu(X) - \nu(A^c) \geq \nu(X) - \mu^*(A^c) = \mu^*(X) - \mu^*(A^c) = \mu^*(A)$, which completes the proof. □

Consider now:

$$\Delta := \{\text{outer measures on } X\} = \{\alpha : \mathcal{P}(X) \to [0,\infty] \mid \alpha \text{ is an outer measure}\}$$

$$\Lambda := \{\text{measures on } X\} = \{(\mathcal{M},\mu) \mid (X,\mathcal{M},\mu) \text{ is a measure space}\},$$

and maps

$$\Delta \ni \alpha \longmapsto (\mathcal{M}_\alpha = \alpha\text{-measurable sets},\ \alpha_c := \alpha|_{\mathcal{M}_\alpha}) \in \Lambda$$

$$\Lambda \ni (\mathcal{M},\mu) \longmapsto \mu^\circ := \mu^* \in \Delta \quad \text{(Caratheodory construction)}.$$

<u>Thm</u>. With the above notation, we have:

1) $(\alpha_c)^\circ = \alpha \iff \alpha$ is regular.

2) $(\mu^\circ)_c = \mu \iff \exists \gamma \in \Delta$: γ is regular and $\mu = \gamma_c$.

3) If μ is complete and σ-finite, then $(\mu^\circ)_c = \mu$.

4) For every $\mu \in \Lambda$, $((\mu^\circ)_c)^\circ = \mu^\circ$.

Pf.: <u>Exercise</u>.

<u>Example</u>: In general, we do <u>not</u> have $((\alpha_c)^\circ)_c = \alpha_c$, for $\alpha \in \Delta$.

Indeed, consider $X = \mathbb{Z}_+$ and $\alpha : \mathcal{P}(X) \to [0,\infty]$ given as $\alpha(Y) := \begin{cases} 0, & Y = \emptyset \\ \sup Y, & Y \neq \emptyset. \end{cases}$

Then α is clearly an outer measure on X, but the only α-measurable sets are $\emptyset$ and X. For if $A \subsetneq X$, $A \neq \emptyset$, then $\exists m,n \in X$ s.t. $m \in A \wedge n \notin A$. Set $E := \{m,n\}$. Then, $\alpha(E) = \max\{m,n\}$ but $\alpha(E\cap A) + \alpha(E \cap A^c) = m+n > \alpha(E)$, so A fails the Caratheodory condition.

Thus, $(\mathcal{M}_\alpha, \alpha_c) = (\{\emptyset, X\}, \alpha_c)$, where $\alpha_c(\emptyset) = 0$, $\alpha_c(X) = \infty$.

Consequently, $(\alpha_c)^\circ(Y) = \infty$ for any $Y \subset X$, $Y \neq \emptyset$. In particular, <u>all</u> subsets of X are $(\alpha_c)^\circ$ measurable and $((\alpha_c)^\circ)_c = (\mathcal{P}(X), \{\emptyset \mapsto 0;\ Y \mapsto \infty, \forall Y \neq \emptyset\})$. □

Let (X,d) be a metric space. For $Y, Z \subset X$, define

$$dist(Y,Z) := \inf d|_{Y\times Z}.$$

<u>Def.</u> Let μ^* be an outer measure on (X,d). We say that μ^* is a <u>metric</u> outer measure when $\mu^*(Y\cup Z) = \mu^*(Y) + \mu^*(Z)$ for any $Y, Z \subset X$ with $dist(Y,Z) > 0$.

<u>NB</u>. $dist(Y,\emptyset) = \infty$, $\forall Y \subset X$.

Thm. Let (X,d) be a metric space, μ^* an outer measure on X, $\mathcal{M}$ the σ-alg. of μ^*-measurable sets.

The following conditions are equivalent:

(1) μ^* is a metric outer measure

(2) $\mathcal{B}(X) \subset \mathcal{M}$.

Pf. (2) $\Rightarrow$ (1): Pick $Y, Z \subset X$ with $\operatorname{dist}(Y,Z) > 0$. We may assume that $Y \neq \emptyset \neq Z$, for else there is nothing to show. Choose ε s.t. $0 < \varepsilon < \operatorname{dist}(Y,Z)$, and let $\Omega := \bigcup_{y \in Y} B_d(y, \frac{\varepsilon}{2})$. Then $\Omega \in \mathcal{B}(X)$, as an open set, so it satisfies the Carathéodory condition:

$$\mu^*(Y \cup Z) = \mu^*(\underbrace{(Y\cup Z) \cap \Omega}_{=Y}) + \mu^*(\underbrace{(Y \cup Z) \cap \Omega^c}_{=Z}) = \mu^*(Y) + \mu^*(Z). \quad \checkmark$$

(1) $\Rightarrow$ (2): It suffices to show that $\mathcal{M}$ contains all the open sets in X.

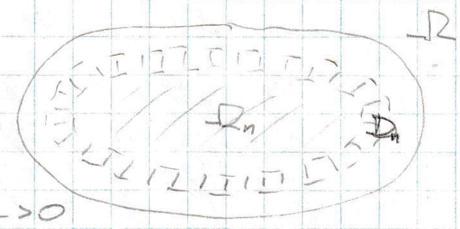
Pick $\Omega \subsetneq X$ open, $\Omega \neq \emptyset$ (for $\emptyset$ or X there is nothing to show).

Pick $E \subset X$ with $\mu^*(E) < \infty$, to check the Carathéodory condition.

For $n \in \mathbb{Z}_+$, define:

$$\Omega_n := \{x \in \Omega : \operatorname{dist}(\{x\}, \Omega^c) > \tfrac{1}{n}\}$$

$$D_n := \{x \in \Omega : \tfrac{1}{n+1} < \operatorname{dist}(\{x\}, \Omega^c) \leq \tfrac{1}{n}\}.$$



We have: $\Omega \setminus \Omega_n = \bigcup_{i=n}^{\infty} D_i$ and $\operatorname{dist}(D_i, D_j) \geq \frac{1}{i+1} - \frac{1}{j} > 0$ for $j > i+1$.

Hence, by assumption (1), $\mu^*(E \cap D_1) + \mu^*(E \cap D_3) + \dots + \mu^*(E \cap D_{2n+1}) \leq \mu^*(E)$

and $\mu^*(E \cap D_2) + \mu^*(E \cap D_4) + \dots + \mu^*(E \cap D_{2n}) \leq \mu^*(E)$ for all $n \in \mathbb{Z}_+$.

Consequently, $\forall n \in \mathbb{Z}_+$, $\sum_{i=1}^{n} \mu^*(E \cap D_i) \leq 2\mu^*(E)$, and so $\sum_{i=1}^{\infty} \mu^*(E \cap D_i) \leq 2\mu^*(E)$.

Since $\mu^*(E) < \infty$, it follows that the series $\sum \mu^*(E \cap D_i)$ converges, and thus $\sum_{i=n}^{\infty} \mu^*(E \cap D_i) \xrightarrow[n \to \infty]{} 0$.

Note that $\mu^*(E) \overset{\text{monotonicity}}{\geq} \mu^*(E \cap (\Omega_n \cup \Omega^c)) = \mu^*((E \cap \Omega_n) \cup (E \cap \Omega^c)) \overset{\text{by (1)}}{=} \mu^*(E \cap \Omega_n) + \mu^*(E \cap \Omega^c)$. (*)

Therefore,

$$\mu^*(E \cap \Omega) + \mu^*(E \cap \Omega^c) \overset{\text{monotonicity}}{\leq} \left(\mu^*(E \cap \Omega_n) + \mu^*(E \cap (\Omega \setminus \Omega_n))\right) + \mu^*(E \cap \Omega^c) \overset{\text{by (*) + monotonicity}}{\leq} \mu^*(E) + \underbrace{\sum_{i=n}^{\infty} \mu^*(E \cap D_i)}_{\downarrow\, n \to \infty \atop 0},$$

which proves that $\mu^*(E \cap \Omega) + \mu^*(E \cap \Omega^c) \leq \mu^*(E)$ $\square$