IV. LEBESGUE MEASURE IN $\mathbb{R}^n$

We will define the Lebesgue outer measure on $\mathbb{R}^n$ by means of Carath. Ext. Theorem.

Fix $n \in \mathbb{Z}_+$.

Def. A <u>box</u> (or an <u>n-box</u>) in $\mathbb{R}^n$ is any set of the form $B = I^1 \times \ldots \times I^n$, where $I^1, \ldots, I^n \subset \mathbb{R}$ are intervals. A 2-box is also called a <u>rectangle</u>.

Consider $\mathcal{A} := \{$finite disjoint unions of boxes in $\mathbb{R}^n\} =$

$$= \Big\{ \bigcup_{j \in J} B_j \;\Big|\; J \text{ is finite},\ B_j = I_j^1 \times \ldots \times I_j^n \text{ for all } j \in J,\ B_k \cap B_l = \emptyset \text{ for } k \neq l \Big\}.$$

<u>Lemma.</u> $\mathcal{A}$ forms an algebra of subsets of $\mathbb{R}^n$.

Pf. Since an n-box $I^1 \times \ldots \times I^{n-1} \times I^n$ is $(I^1 \times \ldots \times I^{n-1}) \times I^n$, it suffices to consider the case <u>$n=2$</u>.

(i) We have $\emptyset = \bigcup_{j \in \emptyset} B_j$, so $\emptyset \in \mathcal{A}$.

Next, observe that the intersection of two (and hence any finite number) of n-boxes is either $\emptyset$ or an n-box.

(ii) Let then $A \in \mathcal{A}$ be arbitrary. If $A = \emptyset$, then $A^c = \mathbb{R}^2 \setminus \emptyset = (-\infty, \infty) \times (-\infty, \infty) \in \mathcal{A}$.
Suppose then that $A = \bigsqcup_{i=1}^{k} I_i^1 \times I_i^2$ for some intervals $I_1^1, I_1^2, \ldots, I_k^1, I_k^2$ in $\mathbb{R}$.
Then, $A^c = \left(\bigsqcup_{i=1}^{k} I_i^1 \times I_i^2 \right)^c = \bigcap_{i=1}^{k} (I_i^1 \times I_i^2)^c = \bigcap_{i=1}^{k} \left((I_i^1)^c \times \mathbb{R} \sqcup I_i^1 \times (I_i^2)^c \right)$, and the latter is a disjoint union of rectangles by the above observation. So $A^c \in \mathcal{A}$.

(iii) Now, to prove that $\mathcal{A}$ is closed under finite unions, it suffices to show that $A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \cap A_2 \in \mathcal{A}$. But this again reduces to the same argument as in (ii). □

Def. Given $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^n)$ as above, define a function $m : \mathcal{A} \to [0, \infty]$ by setting $m(\emptyset) = 0$, $m(I^1 \times \ldots \times I^n) = \text{length}(I^1) \cdots \text{length}(I^n)$, and $m\left(\bigsqcup_{i=1}^{k} B_i\right) = \sum_{i=1}^{k} m(B_i)$.

<u>Convention</u>: $0 \cdot \infty = 0$.

Prop. The function m is a premeasure on $\mathcal{A}$.

Pf. We have $m(\emptyset) = 0$, by definition. Suppose then that $\{B_i\}_{i=1}^{\infty} \subset \mathcal{A}$ is such that $B_i \cap B_j = \emptyset$ $\forall i \neq j$, and $B := \bigcup_{i=1}^{\infty} B_i \in \mathcal{A}$. We need to show that $m(B) = \sum_{i=1}^{\infty} m(B_i)$.
First, note that, for any $k \in \mathbb{Z}_+$, $B \setminus \bigcup_{i=1}^{k} B_i$ is a finite disjoint union of n-boxes, and hence $B = \left(\bigsqcup_{i=1}^{k} B_i \right) \sqcup \left(\bigsqcup_{j=1}^{l} A_j \right)$ for some $A_1, \ldots, A_l \in \mathcal{A}$. Consequently, by def'n of m,
$m(B) = \sum_{i=1}^{k} m(B_i) + \sum_{j=1}^{l} m(A_j)$, hence $\sum_{i=1}^{k} m(B_i) \le m(B)$. Since k was arbitrary, we get $\sum_{i=1}^{\infty} m(B_i) \le m(B)$.

For the opposite inequality, we may assume that $m(B_k)<\infty$, $\forall k$, for else there is nothing to prove. Suppose first that B is compact. we further assume that each B_i is actually an n-box itself, since m is additive on its box compts.

Let $\varepsilon>0$ be arbitrary. For each $i\in\mathbb{Z}_+$ with $m(B_i)>0$ (!), choose $\tilde{B}_i$ an open n-box s.t. $B_i\subset\tilde{B}_i$ and $m(\tilde{B}_i)\le m(B_i)+\varepsilon/2^i$. Then, $\{\tilde{B}_i\}_{i=1}^\infty$ is an open covering of B, so $\exists\, i_1,\dots,i_s$ s.t. $B\subset\tilde{B}_{i_1}\cup\dots\cup\tilde{B}_{i_s}$. As an element of algebra $\mathcal{A}$, the $\tilde{B}_{i_1}\cup\dots\cup\tilde{B}_{i_s}$ can be represented as $\bigsqcup_{j=1}^{\ell}A_j$ for some n-boxes $A_1,\dots,A_\ell$, with $A_j\subset\tilde{B}_{i_j}$ for some $i_j\in\{i_1,\dots,i_s\}$. Also, $(\tilde{B}_{i_1}\cup\dots\cup\tilde{B}_{i_s})\setminus B$ is a union of finitely many n-boxes, and so by def'n of m,

$$m(B)\le m(\tilde{B}_{i_1}\cup\dots\cup\tilde{B}_{i_s})=m\Big(\bigsqcup_{j=1}^{\ell}A_j\Big)=\sum_{j=1}^{\ell}m(A_j)\le\left\{\begin{matrix}\text{since, }\forall\alpha,\\ B_\alpha\setminus\bigsqcup_{A_j\subset B_\alpha}A_j\in\mathcal{A}\end{matrix}\right\}\le m(\tilde{B}_{i_1})+\dots+m(\tilde{B}_{i_s})\le\sum_{i=1}^{\infty}m(B_i)+\varepsilon$$

Since $\varepsilon>0$ was arbitrary, we get $m(B)\le\sum_i m(B_i)$.

Finally, let B be arbitrary. Write $B=\bigsqcup_{j=1}^{p}C_j$, and let $C_1,\dots,C_q$ ($q\le p$) be n-boxes w/ $m(C_j)>0$. Then, we may choose a nested sequence of finite disjoint unions of compact n-boxes $\{A_k\}_{k=1}^\infty$ such that $A_k\uparrow\bigsqcup_{j=1}^{q}C_j$. For each $k\in\mathbb{Z}_+$, we have $m(A_k)\le\sum_{i=1}^\infty m(B_i)$, by above, and hence $m(B)\underset{=m(\bigsqcup_{j=1}^{p}C_j)}{=}\lim_{k\to\infty}m(A_k)\le\sum_{i=1}^\infty m(B_i)$. □

Def. Let $\mathcal{A}\subset\mathcal{P}(\mathbb{R}^n)$ and m be as above. The outer measure m^* on $\mathcal{P}(\mathbb{R}^n)$ defined as $m^*(Y)=\inf\{\sum_{i=1}^\infty m(B_i):\{B_i\}_{i=1}^\infty\subset\mathcal{A},\ Y\subset\bigcup_{i=1}^{\infty}B_i\}$, for $Y\subset\mathbb{R}^n$, is called the <u>Lebesgue outer measure</u> on $\mathbb{R}^n$.

The σ-algebra of m^*-measurable sets is called the σ-alg. of <u>Lebesgue measurable sets</u>, and will be denoted by $\mathcal{L}^n$. The restriction $m^*|_{\mathcal{L}^n}$ is called the <u>Lebesgue measure</u> on $\mathbb{R}^n$, and will be denoted by m (or m^n, if $n=\dim\mathbb{R}^n$ is ambiguous).

<u>**Thm.**</u> Let $n\in\mathbb{Z}_+$, and let m^*, $\mathcal{L}^n$, and m be as above. Then:

(1) m is a complete measure on $\mathcal{L}^n$.
(2) m^* is a metric, regular outer measure on $\mathbb{R}^n$.
(3) $\mathcal{B}(\mathbb{R}^n)\subset\mathcal{L}^n$.
(4) For every n-box B in $\mathbb{R}^n$, $m(B)=\mathrm{vol}(B)$.
(5) For every bounded set $Y\subset\mathbb{R}^n$, $m^*(Y)<\infty$.
(6) m is σ-finite.
(7) $\forall Y\subset\mathbb{R}^n\ \exists G\subset\mathbb{R}^n$ s.t. G is G_δ, $Y\subset G$, and $m^*(Y)=m^*(G)$.
(8) $\forall Y\subset\mathbb{R}^n\ \forall v\in\mathbb{R}^n$: $m^*(Y+v)=m^*(Y)$. (where $Y+v=\{y+v:y\in Y\}$)
(9) $\forall Y\subset\mathbb{R}^n\ \forall s\in[0,\infty]$: $m^*(s\cdot Y)=s^n\,m^*(Y)$. (where $s\cdot Y=\{sy:y\in Y\}$)

Pf. (1) follows from the theorem preceding Caratheodory's Extension Theorem. ✓

(2) That m^* is a regular outer measure follows from — " —

As for "metric", pick $Y, Z \subset \mathbb{R}^n$ s.t. $\mathrm{dist}(Y,Z) =: \varepsilon > 0$. Let $\{B_i\}_{i=1}^{\infty}$ be any countable collection of n-boxes s.t. $Y \cup Z \subset \bigcup_{i=1}^{\infty} B_i$. Note that, $\forall i \in \mathbb{Z}_+$, $B_i = \coprod_{j=1}^{\infty} B_{ij}$ for some countable collection of n-boxes $\{B_{ij}\}$ satisfying $\mathrm{diam}(B_{ij}) < \varepsilon/3$.

Now, $\{B_{ij}\}_{i,j=1}^{\infty} = \underbrace{\{B_{ij} : B_{ij} \cap Z = \phi\}}_{\beta_Y} \sqcup \underbrace{\{B_{ij} : B_{ij} \cap Y = \phi\}}_{\beta_Z}$, as $\mathrm{dist}(Y,Z) > \frac{2\varepsilon}{3}$.

Hence, by def^n of m^*, $m^*(Y) \le \sum_{B_{ij} \in \beta_Y} m(B_{ij})$ and $m^*(Z) \le \sum_{B_{ij} \in \beta_Z} m(B_{ij})$, so

$m^*(Y) + m^*(Z) \le \sum_{i,j=1}^{\infty} m(B_{ij}) = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} m(B_{ij}) \right) = \sum_{i=1}^{\infty} m(B_i)$, b/c the B_{ij} are pairwise disj. for a given i, and m is a premeasure.

Since the collection $\{B_i\}$ was arbitrary covering $Y \cup Z$,

passing to infimum we get $m^*(Y) + m^*(Z) \le m^*(Y \cup Z)$.

The converse inequality follows from monotonicity of m^*. ✓

(3) follows from (2) and equivalence of metricity and containing Borel sets. ✓

(4) follows via Caratheodory's Ext. Thm. from the fact that m is a premeasure on A and $m^*|_A \equiv m|_A$. ✓

(5) follows from monotonicity and the fact that

Y bounded $\Rightarrow \exists k \in \mathbb{Z}_+ : Y \subset [-k,k]^n$, and so $m^*(Y) \le (2k)^n$ ✓

(6) follows from $\mathbb{R}^n = \bigcup_{k \in \mathbb{Z}_+} [-k,k]^n$. ✓

(7) Pick $Y \subset \mathbb{R}^n$ s.t. $m^*(Y) < \infty$. (If $m^*(Y) = \infty$, then take $G = \mathbb{R}^n$ - open, hence G_δ.)

For every $k \in \mathbb{Z}_+$, choose a collection $\{B_{ki}\}_{i=1}^{\infty}$ of n-boxes s.t. $Y \subset \bigcup_{i=1}^{\infty} B_{ki}$ and $\sum_{i=1}^{\infty} m(B_{ki}) < m^*(Y) + \frac{1}{2k}$.

Then, for each B_{ki} choose an open n-box $\widetilde{B}_{ki}$ s.t. $B_{ki} \subset \widetilde{B}_{ki}$ and $m(\widetilde{B}_{ki}) \le m(B_{ki}) + \frac{1}{2k \cdot 2^i}$,

and set $\widetilde{B}_k := \bigcup_{i=1}^{\infty} \widetilde{B}_{ki}$. By construction, $\widetilde{B}_k$ is open, $Y \subset \widetilde{B}_k$, and

$m^*(\widetilde{B}_k) = m(\widetilde{B}_k) \le \sum_{i=1}^{\infty} m(\widetilde{B}_{ki}) \le \sum_{i=1}^{\infty} \left(m(B_{ki}) + \frac{1}{2k \cdot 2^i} \right) = \left(\sum_{i=1}^{\infty} m(B_{ki}) \right) + \frac{1}{2k} \le m^*(Y) + \frac{1}{k}$.

Set $G := \bigcap_{k=1}^{\infty} \widetilde{B}_k$. Then, $Y \subset G$, G is G_δ, and $m^*(G) \overset{\text{by (3)}}{=} m(G) = \lim_{K \to \infty} m\left(\bigcap_{k=1}^{K} \widetilde{B}_k \right) \le$

$\le \lim_{K \to \infty} m(\widetilde{B}_K) \le m^*(Y)$. ✓

(8) follows from the fact that: $\forall$ n-box $B \subset \mathbb{R}^n$, $\mathrm{vol}(B + v) = \mathrm{vol}(B)$. ✓

(9) follows from: $\forall$ n-box $B \subset \mathbb{R}^n$ $\forall s \in [0, \infty]$: $\mathrm{vol}(s \cdot B) = s^n \cdot \mathrm{vol}(B)$. □

Thm. (*Characterization of $\mathcal{L}^n$*) Let $n \in \mathbb{Z}_+$ and let $A \subset \mathbb{R}^n$. Then TFAE:

(1) $A \in \mathcal{L}^n$.

(2) $\forall \varepsilon > 0$ $\exists$ open U s.t. $A \subset U \wedge m^*(U \setminus A) < \varepsilon$.

(3) $\exists$ a G_δ-set B $\exists C$ with $m^*(C) = 0$ s.t. $A = B \setminus C$.

(4) $\forall \varepsilon > 0$ $\exists$ closed F s.t. $F \subset A \wedge m^*(A \setminus F) < \varepsilon$.

(5) $\exists$ an F_σ-set D $\exists C$ with $m^*(C) = 0$ s.t. $A = D \cup C$.

(6) $\exists$ σ-compact K $\exists C$ with $m^*(C) = 0$ s.t. $A = K \cup C$.

Pf:

$$\begin{tikzcd} & (2) \arrow[r, Rightarrow] & (3) \arrow[dl, Rightarrow] & \\ & (1) \arrow[u, Rightarrow] \arrow[dr, Rightarrow] & & \\ (6) \arrow[ur, Rightarrow] & & (4) \arrow[dl, Rightarrow] \\ & (5) \arrow[ul, Rightarrow] & & \end{tikzcd}$$

(1) $\Rightarrow$ (2): Fix $\varepsilon > 0$. Taking intersections with $[-k,k]^n$, we can write A as $\bigcup_{k=1}^\infty A_k$ with $m^*(A_k) < \infty$. Now, for every $k \in \mathbb{Z}_+$, as in the proof of the preceding theorem, we can choose an open U_k satisfying $A_k \subset U_k$ and $m(U_k) \le m^*(A_k) + \varepsilon/2^{k+1}$. Set $U := \bigcup_{k=1}^\infty U_k$.

Then U is open, $A \subset U$, and $U \setminus A = \left(\bigcup_{k=1}^\infty U_k\right) \setminus \left(\bigcup_{k=1}^\infty A_k\right) \subset \bigcup_{k=1}^\infty (U_k \setminus A_k)$, hence

$m^*(U \setminus A) \le \sum_{k=1}^\infty m^*(U_k \setminus A_k) \underset{\substack{b/c\ A_k \\ \text{measurable}}}{\overset{!}{=}} \sum_{k=1}^\infty (m(U_k) - m(A_k)) \le \sum_{k=1}^\infty \varepsilon/2^{k+1} < \varepsilon$. ✓

(2) $\Rightarrow$ (3): For every $k \in \mathbb{Z}_+$, choose U_k open and s.t. $A \subset U_k$, $m^*(U_k \setminus A) < \frac{1}{k}$.
Set $B := \bigcap_k U_k$. Then $A \subset B$, B is G_δ, and $C := B \setminus A$ satisfies $C \subset U_k \setminus A$, $\forall k$, hence $m^*(C) \le \frac{1}{k}$, $\forall k \in \mathbb{Z}_+$. Thus, $m^*(C) = 0$. ✓

(3) $\Rightarrow$ (1): Since $\mathcal{B}(\mathbb{R}^n) \subset \mathcal{L}^n$ and $m = m^*|_{\mathcal{L}^n}$ is complete, it follows that $B \setminus C \in \mathcal{L}^n$. ✓

(1) $\Rightarrow$ (4): Fix $\varepsilon > 0$. By equivalence of (1) and (2), for the Lebesgue measurable set A^c, $\exists U$ open s.t. $A^c \subset U$ and $m^*(U \setminus A^c) < \varepsilon$. Then $F := U^c$ is closed, $F \subset A$, and $A \setminus F = A \cap F^c = A \cap U = U \cap (A^c)^c = U \setminus A^c$, so $m^*(A \setminus F) = m^*(U \setminus A^c) < \varepsilon$. ✓

(4) $\Rightarrow$ (5): For every $k \in \mathbb{Z}_+$, choose F_k closed and s.t. $F_k \subset A$ and $m^*(A \setminus F_k) < \frac{1}{k}$.
Set $D := \bigcup_{k=1}^\infty F_k$. Then, $D \subset A$, D is F_σ, and $C := A \setminus D$ satisfies $C \subset A \setminus F_k$, $\forall k \ge 1$, so $m^*(C) \le m^*(A \setminus F_k) < \frac{1}{k}$, $\forall k \in \mathbb{Z}_+$. Thus, $m^*(C) = 0$. ✓

(5) $\Rightarrow$ (6): Follows from the fact that in $\mathbb{R}^n$ closed sets are σ-compact. ✓

(6) $\Rightarrow$ (1): Follows from the fact that in $\mathbb{R}^n$ σ-compact sets are Borel measurable and $m = m^*|_{\mathcal{L}^n}$ is complete. $\square$

Example (Vitali): $\mathcal{L}^n \neq \mathcal{P}(\mathbb{R}^n)$.

In $\mathbb{R}$, consider the following equivalence relation: $x \sim y :\Leftrightarrow x-y \in \mathbb{Q}$.

Choose $V \subset [0,1]$ any set satisfying: $\forall x \in \mathbb{R}$, $[x]_\sim \cap V \neq \phi$ and $|[x]_\sim \cap V| = 1$.
(Axiom of Choice)

Claim: $V \notin \mathcal{L}^1$.

Suppose otherwise. Then, by translation invariance of m, we have $m(V+r) = m(V)$, for every $r \in \mathbb{Q} \cap [-1,1]$. Since $\mathbb{Q} \cap [-1,1]$ is countable, the union $\bigcup_{r \in \mathbb{Q}\cap[-1,1]} (V+r)$ is in $\mathcal{L}^1$.
Note that, $\forall r, r' \in \mathbb{Q} \cap [-1,1]$, $r \neq r'$: $(V+r) \cap (V+r') = \phi$.
Hence, $m\left(\bigcup_{r\in\mathbb{Q}\cap[-1,1]} (V+r)\right) = \sum_r m(V+r) = \sum_r m(V)$. Thus, $m\left(\bigcup_r (V+r)\right)$ is either 0 or ∞. But, we have $[0,1] \subset \bigcup_{r \in \mathbb{Q}\cap[-1,1]} (V+r) \subset [-1,2]$, so $1 \leq m\left(\bigcup_r (V+r)\right) \leq 3$. ↯

$\forall x \in [0,1]$ $\exists y \in V \subset [0,1]$ s.t. $x - y =: r \in \mathbb{Q}$; but then $r \in [-1,1]$

b/c $V \subset [0,1]$

Uniqueness of Lebesgue Measure

Prop. Let $n \in \mathbb{Z}_+$, let m^* be the Lebesgue outer measure on $\mathbb{R}^n$, and let $\mathcal{L}^n$ be the σ-alg of Lebesgue measurable sets. Suppose that $\mathcal{M}$ is a σ-algebra in $\mathbb{R}^n$ s.t. $\mathcal{M}$ contains (compact) n-boxes and $m^*|_{\mathcal{M}}$ is a measure. Then, $\mathcal{M} \subset \mathcal{L}^n$.

Pf. Note that $\mathcal{B}(\mathbb{R}^n) \subset \mathcal{M}$, since $\mathcal{M}$ contains all open n-boxes.
Pick $Y \in \mathcal{M}$. To show that Y is m^*-measurable, pick any test set $E \subset \mathbb{R}^n$.
By the Basic Properties of Leb. Measure Theorem, can choose a G_δ-set $G \subset \mathbb{R}^n$ s.t. $m^*(E) = m^*(G)$ and $E \subset G$. Hence,

$$m^*(E\cap Y) + m^*(E \cap Y^c) \leq m^*(G \cap Y) + m^*(G \cap Y^c) \overset{G,Y\in\mathcal{M} \text{ and } m^*|_{\mathcal{M}} \text{ is additive}}{=} m^*(G) = m^*(E),$$ i.e., $Y \in \mathcal{L}^n$. $\square$

Prop. Suppose that μ is a measure on the σ-alg. $\mathcal{L}^n$ s.t. $\mu(B) = vol(B)$ for every (compact) n-box B. Then, $\mu = m$.

Pf. By assumption, μ is an extension of the premeasure m on the algebra $\mathcal{A}$ of finite disjoint unions of n-boxes in $\mathbb{R}^n$ (since every n-box is a denumerable union of compact n-boxes).
Let $A \in \mathcal{L}^n$ be arbitrary. For any collection of n-boxes $\{B_k\}_{k=1}^\infty$ with $A \subset \bigcup_{k=1}^\infty B_k$, we have

$\mu(A) \le \mu(\bigcup_k B_k) \le \sum_k \mu(B_k) = \sum_k \mathrm{vol}(B_k) = \sum_k m(B_k)$. Passing to inf over all such collections, we get $\mu(A) \le m(A)$.

Conversely, to show $\mu(A) \ge m(A)$, suppose first that A is bounded; i.e, $\exists k \in \mathbb{Z}_+$ st. $A \subset [-k,k]^n$

Then, by above, $(2k)^n - \mu(A) = \mu([-k,k]^n \setminus A) \le m([-k,k]^n \setminus A) = (2k)^n - m(A)$, so $\mu(A) \ge m(A)$.

Finally, an arbitrary A may be realized as $A = \coprod_{k=1}^{\infty} A_k$ with A_k bounded, and hence

$\mu(A) = \sum_k \mu(A_k) \ge \sum_k m(A_k) = m(A)$. □

Lebesgue-Stieltjes Measures

In $\mathbb{R}$, consider as before $\mathcal{A} = \{$ finite disjoint unions of intervals $\} \subset \mathcal{P}(\mathbb{R})$.

Let $\alpha : \mathbb{R} \to \mathbb{R}$ be an increasing right-continuous function; i.e, $\alpha(x) \le \alpha(y)$ $\forall x < y$, and $\alpha(x) = \lim_{y \to x^+} \alpha(y)$ $\forall x$. For an interval $I \subset \mathbb{R}$ with endpoints $a < b$, define $\ell(I) := \alpha(b) - \alpha(a)$. If $a = -\infty$, set $\ell(I) := \alpha(b) - \lim_{x \to -\infty} \alpha(x)$; if $b = \infty$, $\ell(I) := \lim_{x \to \infty} \alpha(x) - \alpha(a)$

Example: $\alpha = \mathrm{id}$.

Prop. The function m defined on $\mathcal{A}$ as $m(\coprod_{k=1}^{K} I_k) := \sum_{k=1}^{K} \ell(I_k)$ is a premeasure.

Pf: Exercise!
Hint: Adapt the proof for Lebesgue premeasure. Note that, if I is a bdd interval with endpoints a, b, then $I \subset \{a\} \cup (a, b+\delta)$, where $\forall \varepsilon > 0$ δ can be chosen so that $\ell((a, b+\delta)) \le \ell(I) + \varepsilon$.

Def. The outer measure resulting from Caratheodory's Extension of m to $\mathcal{P}(\mathbb{R})$ is called a Lebesgue-Stieltjes outer measure (rel. to α). The restriction of this m^* to m^*-measurable sets is called a Lebesgue-Stieltjes measure (rel. to α)

V. HAUSDORFF MEASURE.

Let (X, d) be a metric space, $p \in [0, \infty)$.

Def. For a set $Y \subset X$, define $h^p(Y) := \begin{cases} 2^{-p} \cdot \alpha(p) \cdot (\mathrm{diam}\, Y)^p, & Y \ne \emptyset \\ 0, & Y = \emptyset \end{cases}$

where $\alpha(p) = \dfrac{(\Gamma(\frac{1}{2}))^p}{\Gamma(\frac{p}{2}+1)}$ and $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$. /NB. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$/ (Euler's Γ function)

Def. Given (X,d), $p\in[0,\infty)$, and $h^p : \mathcal{P}(X)\to[0,\infty]$ as above, the <u>p-dimensional Hausdorff outer measure</u> is defined as follows

$$h^{p*}(Y) := \sup_{\delta>0}\,\inf\Big\{\sum_{k=1}^{\infty} h^p(Y_k) \;\Big|\; \{Y_k\}_{k=1}^{\infty}\subset\mathcal{P}(X),\ Y\subset\bigcup_{k=1}^{\infty}Y_k,\ \operatorname{diam}Y_k<\delta\Big\}\ \Big(=\lim_{\delta\to0^+}\inf\{\ \}\Big)$$

<u>Convention:</u> $0^0=\infty^0=1$, and $0^p=0$, $\infty^p=\infty$ for $p>0$.

<u>Def.</u> Given $p\in[0,\infty)$, the σ-alg. of h^{p*}-measurable sets is called the <u>Hausdorff p-measurable sets</u>, and will be denoted by $\mathcal{H}^p$. We'll denote $h^{p*}|_{\mathcal{H}^p}$ by h^p again.

<u>Remarks</u>: One can show that

1) In $X=\mathbb{R}^3$, $h^0=$ counting measure (in fact, in any metric space (X,d))
   h^1 measures lengths of curves
   h^2 —"— areas
   $h^3=m^3$
2) In $\mathbb{R}^n$ (with Euclidean metric), $h^{n*}=m^{n*}$.
3) In any metric space (X,d), $\forall p\in[0,\infty)$, we have:
   (a) h^{p*} is a metric regular outer measure
   (b) h^p is a complete measure
   (c) $\forall Y\subset X$ $\exists$ a G_δ-set G s.t. $Y\subset G \wedge h^{p*}(Y)=h^{p*}(G)$.

   TFAE: (a) $A\subset X$ is h^{p*}-measurable
   (b) $\exists$ G_δ-set B $\exists C$ with $h^{p*}(C)=0$ s.t. $A=B\setminus C$
   (c) $\exists$ F_σ-set D $\exists C$ with $h^{p*}(C)=0$ s.t. $A=D\cup C$.
   } provided $h^{p*}(A)<\infty$

<u>Hausdorff dimension of a metric space</u>

<u>Key Lemma.</u> Let (X,d) be a metric space, and let $s\in(0,\infty)$. Suppose $h^s(X)<\infty$. Then, $h^p(X)=0$ for all $p>s$.

Pf. For a proof by contradiction, suppose $p>s$ is s.t. $h^p(X)>0$; say $h^p(X)\geq\frac{2^p}{\alpha(p)}2\alpha$, with $\alpha>0$. Then, by def'n, $\exists\delta_0>0$ s.t. $\forall\delta\in(0,\delta_0]$ $\forall\{Y_k^\delta\}_{k=1}^\infty$ s.t. $X=\bigcup_{k=1}^\infty Y_k^\delta$ and $\operatorname{diam}Y_k^\delta<\delta$, $\forall k$, we have

$$\sum_{k=1}^\infty(\operatorname{diam}Y_k^\delta)^p\geq\frac{3\alpha}{2}. \quad (\odot)$$

(*) { Note that, $\forall Y\subset X$, if $Y=\bigcup_{i=1}^\infty Z_i$, then $\operatorname{diam}Y\leq\sum_{i=1}^\infty\operatorname{diam}Z_i$ and hence $(\operatorname{diam}Y)^p\leq\sum_{i=1}^\infty(\operatorname{diam}Z_i)^p$, since this is the case for any finite union.

We want to show that $h^s(X)=\infty$.

(**) { Equivalently, we want: $\forall M>0$ $\exists\delta_M>0$ s.t. $\forall\{Y_k\}_{k=1}^\infty$ with $X=\bigcup_{k=1}^\infty Y_k$ and $\operatorname{diam}Y_k<\delta_M$, $\forall k$, we have $\sum_{k=1}^\infty(\operatorname{diam}Y_k)^s\geq M$.

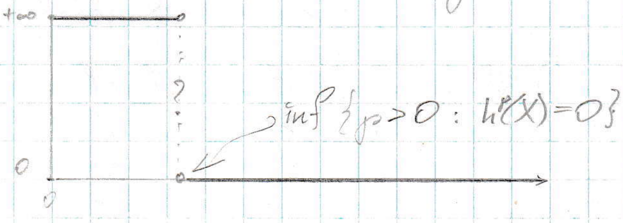
Now, by $(*)$, it suffices to show that $(\overset{*}{*})$ holds for all families $\{Y_k\}$ with

$$\frac{\delta_M}{2} \leq \operatorname{diam} Y_k < \delta_M \quad , \quad \forall k$$

Let then $M>0$ be arbitrary. Choose $\delta_M \in (0, \delta_0]$ such that $\dfrac{\alpha}{2^s \cdot \delta_M^{p-s}} \geq M$.
(Such δ_M exists, b/c $p-s>0 \Rightarrow \dfrac{\alpha}{2^s \cdot \delta^{p-s}} \xrightarrow[\delta \to 0^+]{} \infty$.)
Let now $\{Y_k\}_{k=1}^{\infty} \subset \mathcal{P}(X)$ be any such that $X = \bigcup_{k=1}^{\infty} Y_k$ and $\frac{\delta_M}{2} \leq \operatorname{diam} Y_k < \delta_M$, $\forall k$.
By $(\odot)$, we can choose a $K \in \mathbb{Z}_+$ s.t. $\sum_{k=1}^{K} (\operatorname{diam} Y_k)^p \geq \alpha$.
Then, $K \cdot \delta_M^p > \sum_{k=1}^{K} (\operatorname{diam} Y_k)^p \geq \alpha$, so $K \geq \dfrac{\alpha}{\delta_M^p}$, and hence

$$\sum_{k=1}^{\infty} (\operatorname{diam} Y_k)^s \geq \sum_{k=1}^{K} (\operatorname{diam} Y_k)^s \geq K \cdot \left(\frac{\delta_M}{2}\right)^s \geq \frac{\alpha}{2^s \cdot \delta_M^{p-s}} \geq M \text{, as required. } \square$$

Corollary. The graph of the function $[0,\infty) \ni p \longmapsto h^p(X) \in [0,\infty]$ looks as follows:



Def. Given a non-empty metric space (X,d), the number $\dim_{\mathcal{H}}(X) := \inf\{p>0 : h^p(X)=0\} \in [0,\infty]$ is called the Hausdorff dimension of X.

Corollary. Let (X,d) be a metric space.
(1) If $s < \dim_{\mathcal{H}}(X)$, then $h^s(X) = \infty$.
(2) If $p > \dim_{\mathcal{H}}(X)$, then $h^p(X) = 0$.
(3) If $h^p(X) \in (0,\infty)$, then $\dim_{\mathcal{H}}(X) = p$.

Example: Find the Hausdorff dimension of the Cantor set in $[0,1]$.

Sol'n:
Note that, $\forall k \in \mathbb{Z}_+$, C can be covered by the disjoint union of 2^k closed intervals of length $\frac{1}{3^k}$ each. Hence, $\forall k$, $\inf\left\{ \sum_{i=1}^{\infty} (\operatorname{diam} Y_i)^p \;\middle|\; \{Y_i\}_{i=1}^{\infty} \subset \mathcal{P}([0,1]) \text{ s.t. } C \subset \bigcup_{i=1}^{\infty} Y_i, \operatorname{diam} Y_i \leq \frac{1}{3^k} \right\}$ is less than or equal to $2^k \cdot \frac{1}{3^{pk}}$.

Thus, for any $p>0$, we have $h^p(C) \leq \dfrac{\alpha(p)}{2^p} \cdot \lim_{k\to\infty} \dfrac{2^k}{3^{pk}}$.

Now, $\lim_{k \to \infty} \left(\dfrac{2}{3^p}\right)^k = \begin{cases} 0, & 2 < 3^p \\ 1, & 2 = 3^p \\ \infty, & 2 > 3^p \end{cases}$. Hence, $\dim_{\mathcal{H}}(C) \geq \log_3 2$.

On the other hand, if $p > \log_3 2$, then, $\forall k \geq 1$, $\frac{1}{(3^k)^p} > \frac{2}{(3^{k+1})^p}$.

It follows that for every $k \geq 1$, the k'th iteration C_k of the Cantor set C satisfies

$$\inf\left\{ \sum_i (\operatorname{diam} Y_i)^p \;\middle|\; \{Y_i\}_{i=1}^{\infty} \text{ s.t. } C_k \subset \bigcup_i Y_i,\ \operatorname{diam} Y_i \leq \frac{1}{3^k} \right\} \leq \frac{2^k}{3^{kp}}.$$

Hence, $h^p(C_k) \leq \frac{\alpha(p)}{2^p} \cdot \frac{2^k}{3^{kp}}$.

Now, $C \in \mathcal{B}([0,1])$ and h^p is a metric outer measure, so C and all the C_k are measurable, and hence $h^p(C) = \lim_{k\to\infty} h^p(C_k) \leq \frac{\alpha(p)}{2^p} \lim_{k\to\infty} \frac{2^k}{3^{kp}} = 0$. Thus $\dim_H(C) \leq \log_3 2$. $\square$