VI. MEASURABLE FUNCTIONS

Let $(X, \mathcal{M})$ be a measurable space. Notation: $\overline{\mathbb{R}} := \{-\infty\} \sqcup \mathbb{R} \sqcup \{+\infty\}$

Def. A function $f: X \to \overline{\mathbb{R}}$ is called $\mathcal{M}$-measurable (or simply, measurable), when

$\forall a \in \mathbb{R}, \quad \{x \in X \mid f(x) > a\} \in \mathcal{M}$. (A function $f = (f_1, \ldots, f_m): X \to V$ to an m-dim. real vector space $V \cong \mathbb{R}^m$ is measurable when all f_i are so.)

Examples:

1) Every constant function is measurable.

Indeed, if $f \equiv c$ for some $c \in \overline{\mathbb{R}}$, then $f^{-1}((a, \infty]) = \begin{cases} \emptyset, & c \leq a \\ X, & c > a \end{cases}$

2) Let $f = \chi_A$ for some $A \subset X$. Then f is measurable iff $A \in \mathcal{M}$.

Indeed, given $a \in \mathbb{R}$, we have $f^{-1}((a, \infty]) = \begin{cases} \emptyset & ; a \geq 1 \\ A & ; 0 \leq a < 1 \\ X & ; a < 0 \end{cases}$.

Def. Let $A \in \mathcal{M}$. A function $f: A \to \overline{\mathbb{R}}$ is measurable when

$$\forall a \in \mathbb{R}, \quad \{x \in A \mid f(x) > a\} = A \cap f^{-1}((a, \infty]) \in \mathcal{M}.$$

Prop. Let $(X, \mathcal{M})$ be a measurable space, $A \in \mathcal{M}$, and $f: A \to \overline{\mathbb{R}}$. TFAE:

(1) f is measurable

(2) $\forall a \in \mathbb{R}, \quad \{x \in A \mid f(x) \leq a\} \in \mathcal{M}$

(3) $\forall a \in \mathbb{R}, \quad \{x \in A \mid f(x) < a\} \in \mathcal{M}$

(4) $\forall a \in \mathbb{R}, \quad \{x \in A \mid f(x) \geq a\} \in \mathcal{M}$

(5) $\forall$ interval $I \subset \overline{\mathbb{R}}, \quad f^{-1}(I) \in \mathcal{M}$

(6) $\forall$ open $U \subset \overline{\mathbb{R}}, \quad f^{-1}(U) \in \mathcal{M}$

(7) $\forall B \in \mathcal{B}(\overline{\mathbb{R}}), \quad f^{-1}(B) \in \mathcal{M}$.

Proof = Exercise (!)

Measurable Functions - Addendum:

$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, where $-\infty \neq \infty$.

Linear ordering: $\forall x \in \mathbb{R}$, $-\infty < x < \infty$.

Arithmetics: • $\forall x \in \mathbb{R}$, $-\infty \pm x = -\infty$, $\infty \pm x = \infty$, $\infty + \infty = \infty$, $-\infty + (-\infty) = -\infty$,

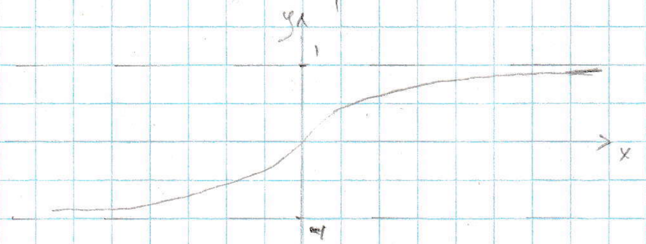
$\infty + (-\infty) = 0$, $-(\infty) = -\infty$, $-(-\infty) = \infty$;

• $|-\infty| = |\infty| = \infty$;

• $\forall x, y \in \overline{\mathbb{R}}$, $x - y := x + (-y)$.

• $(-\infty)\cdot(-\infty) = \infty$, $(-\infty)\cdot\infty = -\infty$, $\forall x > 0$, $-\infty \cdot x = -\infty$, $-\infty\cdot(-x) = \infty$.

Topology: Define $\varphi : \overline{\mathbb{R}} \to [-1,1]$ as $\varphi(x) = \begin{cases} -1, & x = -\infty \\ \frac{x}{1+|x|}, & x \in \mathbb{R} \\ 1, & x = \infty \end{cases}$



Then the function $d(x,y) := |\varphi(x) - \varphi(y)|$ defines a metric on $\overline{\mathbb{R}}$.

We regard $\overline{\mathbb{R}}$ as a top. space with the top. induced by d. It coincides with the top. induced by φ^{-1} from $[-1,1]$ as a subspace of $\mathbb{R}$ with Euclidean top.

Def. A function $f : (X, \mathcal{M}) \to \overline{\mathbb{R}}$ is measurable, when $f^{-1}((a,\infty]) \in \mathcal{M}$, $\forall a \in \mathbb{R}$.

Remark. $f : (X,\mathcal{M}) \to \overline{\mathbb{R}}$ measurable $\Rightarrow f^{-1}(-\infty), f^{-1}(\infty) \in \mathcal{M}$.

Indeed, $f^{-1}(\infty) = \{x \in X \mid \forall n \in \mathbb{Z}_+, f(x) > n\} = \bigcap_{n=1}^{\infty} \{x \mid f(x) > n\} = \bigcap_{n=1}^{\infty} f^{-1}((n,\infty])$

$f^{-1}(-\infty) = \bigcap_{n=1}^{\infty} f^{-1}([-\infty, n))$.

(!) Example: $f : (X, \mathcal{M}) \to \overline{\mathbb{R}}$ measurable $\not\Leftarrow$ $f^{-1}(U) \in \mathcal{M}$, $\forall U$ open in $\mathbb{R}$.

E.g. $f(x) := \begin{cases} \infty, & x \in V \\ -\infty, & x \in \mathbb{R}\setminus V \end{cases}$, $f : \mathbb{R} \to \overline{\mathbb{R}}$.

Corollary. If $f:\overline{\mathbb{R}}\to\overline{\mathbb{R}}$ is continuous, then f is Borel-measurable.

Pf. Indeed, by the prop., it suffices to show that $f^{-1}(U)\in\mathcal{B}(\overline{\mathbb{R}})$ for every open $U\subset\overline{\mathbb{R}}$. But $f^{-1}(U)$ is open, by continuity. □

Prop. Let $(X,\mathcal{M})$ be a measurable space, $A\in\mathcal{M}$, $f:A\to\overline{\mathbb{R}}$ measurable, and $\varphi:\overline{\mathbb{R}}\to\overline{\mathbb{R}}$ Borel-measurable. Then $\varphi\circ f$ is measurable.

Pf. For any open $U\subset\overline{\mathbb{R}}$, $\varphi^{-1}(U)\in\mathcal{B}(\overline{\mathbb{R}})$, and hence $(\varphi\circ f)^{-1}(U)=f^{-1}(\varphi^{-1}(U))\in\mathcal{M}$. □

Prop. Let $(X,\mathcal{M})$ be a measurable space, $A\in\mathcal{M}$, $f:A\to\overline{\mathbb{R}}$. Then:

(a) If f is measurable, then $f|_B:B\to\overline{\mathbb{R}}$ is measurable, $\forall B\in\mathcal{M}, B\subset A$.

(b) If $A=\bigcup_{i=1}^{\infty}A_i$ and $f|_{A_i}$ is measurable for all i, then f is measurable.

Pf. = Exercise.

Prop. Let $(X,\mathcal{M})$ measurable space, $A\in\mathcal{M}$, $f,g:A\to\overline{\mathbb{R}}$ measurable, $c\in\mathbb{R}$.
Then, the functions $c\cdot f$, $-f$, $|f|$, f^2, $f+g$, $f\cdot g$, $\max\{f,g\}$, and $\min\{f,g\}$ are measurable.

Pf. The functions
$$\left.\begin{array}{l}\varphi_1:\overline{\mathbb{R}}\ni x\mapsto c\cdot x\in\overline{\mathbb{R}}\\ \varphi_2:\overline{\mathbb{R}}\ni x\mapsto -x\in\overline{\mathbb{R}}\\ \varphi_3:\overline{\mathbb{R}}\ni x\mapsto |x|\in\overline{\mathbb{R}}\\ \varphi_4:\overline{\mathbb{R}}\ni x\mapsto x^2\in\overline{\mathbb{R}}\end{array}\right\}$$
are continuous and hence Borel measurable.

As for $f+g$, $\forall a\in\mathbb{R}$: $\{x\in A\mid f(x)+g(x)>a\}=\{x\in A\mid f(x)>a-g(x)\}=$
$$=\bigcup_{r\in\mathbb{Q}}\left(\{x\in A\mid f(x)>r\}\cap\{x\in A\mid r>a-g(x)\}\right)\in\mathcal{M}.\ \checkmark$$

Now, $f\cdot g=\frac{1}{4}\left[(f+g)^2-(f-g)^2\right]$.

$(\max\{f,g\})^{-1}((a,\infty])=f^{-1}((a,\infty])\cup g^{-1}((a,\infty])$.

$\min\{f,g\}=-\max\{-f,-g\}$. □

Corollary. If $f,g:A\to\overline{\mathbb{R}}$ are measurable, then the sets
$$\{x\in A:\ f(x)\le g(x)\},\ \{x\in A:\ f(x)>g(x)\},\ \{x\in A:\ f(x)=g(x)\}$$ are measurable.

Pf. Since $g-f$ is measurable, then $\{x\in A: f(x)\le g(x)\}=\{x\in A: (g-f)(x)\ge 0\}\in\mathcal{M}$.
Then, $\{x\in A: f(x)>g(x)\}=A\setminus\{x\in A: f(x)\le g(x)\}\in\mathcal{M}$, and
$$\{x\in A: f(x)=g(x)\}=\{x: f(x)\le g(x)\}\cap\{x: f(x)\ge g(x)\}\in\mathcal{M}.\ \square$$

Prop. Let $A \in \mathcal{M}$ and let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions $f_n : A \to \overline{\mathbb{R}}$.
Then, $\inf_{n\geq 1} f_n$, $\sup_{n\geq 1} f_n$, $\lim_{n\to\infty}\inf f_n$, $\lim_{n\to\infty}\sup f_n$ are measurable.
Moreover, the set $\{x\in A : (f_n(x))_{n=1}^{\infty}$ is convergent$\}$ is measurable, and if (f_n) is pointwise convergent then $\lim_{n\to\infty} f_n$ is measurable.

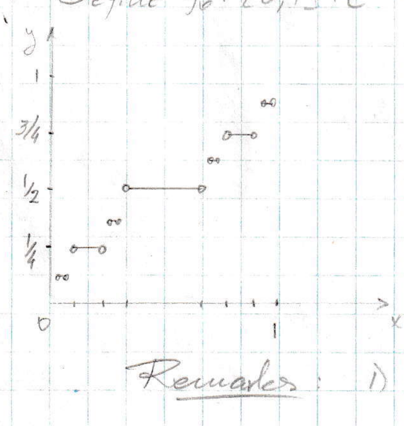
Pf. For any $a\in\mathbb{R}$, we have $\{x\in A : (\inf_{n\geq 1} f_n)(x) \geq a\} = \bigcap_{n\geq 1} \underbrace{\{x\in A : f_n(x)\geq a\}}_{\in\mathcal{M}} \in \mathcal{M}$,
and $\{x\in A : (\sup_{n\geq 1} f_n)(x) > a\} = \bigcup_{n\geq 1} \underbrace{\{x\in A : f_n(x) > a\}}_{\in\mathcal{M}} \in \mathcal{M}$.
Now, $\lim_{n\to\infty}\inf f_n = \sup_{n\geq 1}(\inf_{k\geq n} f_k)$ and $\lim_{n\to\infty}\sup f_n = \inf_{n\geq 1}(\sup_{k\geq n} f_k)$.
Moreover, $\{x\in A : (f_n(x))_n$ converges$\} = \{x\in A : \lim_{n\to\infty}\inf f_n(x) = \lim_{n\to\infty}\sup f_n(x)\} \in \mathcal{M}$, and if $(f_n)_n$ converges ptwise, then $\lim_{n\to\infty} f_n = \lim_{n\to\infty}\inf f_n = \lim_{n\to\infty}\sup f_n$. □

Next, we want to construct an example of a Lebesgue measurable set which is not Borel measurable.

Prop. If $f:\overline{\mathbb{R}}\to\overline{\mathbb{R}}$ is monotone, then f is Borel measurable.

Pf.
Suppose f is decreasing. Given $a\in\mathbb{R}$, set $x_0 := \inf\{x : f(x)\leq a\}$.
Then, $f^{-1}((a,\infty]) = \begin{cases} [-\infty, x_0), & f(x_0) = a \\ [-\infty, x_0], & f(x_0) > a \end{cases}$, so in any case it is in $\mathcal{B}(\overline{\mathbb{R}})$. □

Example:
Let $C\subset[0,1]$ be the ternary Cantor set. Note that the right endpoint of every (maximal) open interval in $[0,1]\setminus C$ is of the form $a = 0.a_1a_2\ldots a_k000\ldots$ with $a_1,\ldots,a_k\in\{0,2\}$.
Define $f_0 : [0,1]\setminus C \to [0,1]$ by setting $f_0(x) = \sum_{i=1}^{k}\frac{a_i}{2^{i+1}}$ for all x in the interval with the right endpt $0.a_1\ldots a_k$.



On $[0,1]$, we define the Cantor-Lebesgue function f by

$$f(x) = \begin{cases} \inf\{f_0(y) \mid y\geq x,\ y\in[0,1]\setminus C\}, & x<1 \\ 1, & x=1 \end{cases}$$

Remarks: 1) $f \equiv f_0$ on $[0,1]\setminus C$.

2) f is increasing. [inf over a bigger set is less than or equal]

3) $f : [0,1]\to[0,1]$

(!) 4) f is continuous. Indeed, as a monotone function, f could have at most jumps discontinuities. Suppose then that $a\in[0,1]$ is st.
$\lim_{x\to a^-} f(x) < \lim_{x\to a^+} f(x)$. Then, $\exists n\in\mathbb{Z}_+$ $\exists k\in\{1,\ldots,2^n-1\}$ st. $\frac{k}{2^n}\notin \text{range}(f)$

But f attains all values of the form $\frac{k}{2^n}$, b/c for any $k\in\{1,\dots,2^n-1\}$ there are $l\in\mathbb{N}$ $(l\le n-1)$ and $\varepsilon_0,\dots,\varepsilon_l\in\{0,1\}$ s.t. $k=\varepsilon_0+\varepsilon_1 2+\dots+\varepsilon_l 2^l$. Thus, f has no jump discontinuities.

Next, define $F:[0,1]\to[0,1]$ by $F(x):=\inf\{y \mid f(y)\ge x\}$.

Remarks: 1) F is strictly increasing (and hence one-to-one).

Indeed, if $S_x=\{y \mid f(y)\ge x\}$ and $x_1<x_2$, then $S_{x_1}\supset S_{x_2}$, so $\inf S_{x_1}\le \inf S_{x_2}$.

If $S_{x_1}=S_{x_2}$, then f would have no values in the interval (x_1,x_2), which contradicts the Intermediate Value Thm., as f is continuous on $[0,1]$, $f(0)=0$ and $f(1)=1$.

2) By 1), $F^{-1}(F(V))=V$, for any $V\subset[0,1]$.

(!) b/c $F([0,1])\subseteq C$

Let then V be a Vitali set in $[0,1]$, and set $B:=F(V)$. Then $B\subset C$, and $m(C)=0$, so $B\in\mathcal{L}^1$ as a null set. But $B\notin\mathcal{B}(\mathbb{R})$, for otherwise $V=F^{-1}(B)$ would be Borel, and hence Lebesgue, measurable. Thus, $\mathcal{B}(\mathbb{R})\subsetneq\mathcal{L}^1$ □

Def. Given $f:A\to\overline{\mathbb{R}}$, define $f^+:=\max\{0,f\}$, $f^-=\max\{0,-f\}$. Then,

$$f=f^+-f^- \quad\text{and}\quad |f|=f^++f^-.$$

Prop. Let $(X,\mathcal{M})$ be a measurable space, $A\in\mathcal{M}$, and $f:A\to\overline{\mathbb{R}}$. Then TFAE:

(1) f is measurable

(2) f^+ and f^- are measurable.

Def. Let $(X,\mathcal{M})$ be a measurable space. A function $s:X\to\overline{\mathbb{R}}$ is called a *simple function* when $s=\sum_{i=1}^n a_i\chi_{A_i}$ for some $n\in\mathbb{Z}_+$, $A_1,\dots,A_n\in\mathcal{M}$ and $a_1,\dots,a_n\in\mathbb{R}$.

Given a function $f:(X,\mathcal{M})\to\overline{\mathbb{R}}$, define for each $n\in\mathbb{Z}_+$

$$s_n(x):=\min\{n,2^{-n}\cdot[2^n\cdot f^+(x)]\}-\min\{n,2^{-n}\cdot[2^n\cdot f^-(x)]\}$$

Convention (!): $[\infty]=\infty$, $[-\infty]=-\infty$

Thm. Let $(X,\mathcal{M})$ be a measurable space, $A\in\mathcal{M}$, and let $f:A\to\overline{\mathbb{R}}$ be measurable. Then

(1) $\{s_n\}_{n=1}^\infty$ are simple functions

(2) $(s_n)_{n=1}^\infty$ converges pointwise to f on A

(3) f nonnegative $\Rightarrow$ s_n nonnegative, $\forall n$

(4) f nonnegative $\Rightarrow$ $(s_n(x))_{n=1}^\infty$ is increasing, $\forall x\in A$

(5) f bounded $\Rightarrow$ $s_n \underset{A}{\rightrightarrows} f$ $(n\to\infty)$

Pf. Suppose first that $f \geq 0$.

Note that the "integral part" function $[\cdot]: \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is increasing, and hence Borel measurable.

It follows that the s_n are measurable. Since each s_n admits at most $1 + n \cdot 2^n$ different values, it must be a simple function. This proves (1). ✓

(2): For every $x \in A_n \cap \{x \mid f(x) < \infty\}$, we have

$$2^n \cdot f(x) - 1 \leq [2^n \cdot f(x)] \leq 2^n f(x),$$

hence $f(x) - \frac{1}{2^n} \leq 2^{-n} \cdot [2^n f(x)] \leq f(x)$, and thus for all $n \gg 1$,

$$f(x) - \frac{1}{2^n} \leq s_n(x) \leq f(x). \quad \text{Therefore } s_n(x) \xrightarrow[n\to\infty]{} f(x). \checkmark$$

(3) is obvious.

(4) For any $x \in A$, we have $[2^{n+1} \cdot f(x)] \geq 2 \cdot [2^n f(x)]$, and hence $2^{-(n+1)}[2^{n+1} f(x)] \geq 2^{-n}[2^n f(x)]$.

This implies that $s_{n+1}(x) \geq s_n(x)$. ✓

(5) Suppose $|f| \leq M$. Then, for $n \gg 1$,

$s_n = 2^{-n} \cdot [2^n f(x)]$, and hence (as in (2)), $f(x) - \frac{1}{2^n} \leq s_n(x) \leq f(x)$, $\underline{\forall x \in A}$.

The choice of n is independent of x, so $s_n \rightrightarrows f$. ✓

Now, for an arbitrary f, we have $f = f^+ - f^-$ and f^+, f^- measurable.

Define $s_n'(x) := \min\{n, 2^{-n}[2^n f^+(x)]\}$ and $s_n''(x) := \min\{n, 2^{-n}[2^n f^-(x)]\}$.

Then $s_n' \xrightarrow[n\to\infty]{} f^+$ and $s_n'' \xrightarrow[n\to\infty]{} f^-$, so $s_n' - s_n'' \xrightarrow[n\to\infty]{} f^+ - f^- = f$.

Finally, if f is bdd, then so are f^+, f^-, so $s_n' \rightrightarrows f^+$ and $s_n'' \rightrightarrows f^-$ □

——— CUT-OFF FOR MIDTERM 1 ———

VII. THE LEBESGUE INTEGRAL.

Def. Let $(X, \mathcal{M}, \mu)$ be a measure space. If $s: X \to [0, \infty)$ is a non-negative simple function, and $s = \sum_{i=1}^{k} \underbrace{a_i}_{\geq 0} \chi_{A_i}$, we define the Lebesgue integral of s as

$$\int s \, d\mu = \int s := \sum_{i=1}^{k} a_i \, \mu(A_i).$$

If $f: X \to [0, \infty]$ is a non-negative measurable function, the Lebesgue integral of f is

$$\int f \, d\mu := \sup\left\{ \int s \, d\mu : s \text{ simple}, \ 0 \leq s \leq f \right\}.$$

If $f: X \to \overline{\mathbb{R}}$ is measurable, and $\int f^+ d\mu$ and $\int f^- d\mu$ are not simultaneously ∞, we define

$$\int f \, d\mu := \int f^+ d\mu - \int f^- d\mu.$$

Finally, if $f: X \to \mathbb{C}$, $f = u + iv$, and $\int |u| + |v| \, d\mu < \infty$, define $\int f \, d\mu = \int u \, d\mu + i \cdot \int v \, d\mu$.

<u>Remarks</u>: 1) Recall that, by our convention, $0\cdot\infty=0$. (In particular, $\int 0\,d\mu=0$.)

2) The integral of a simple function is well-defined; i.e., independent of representation ($\sum_{i=1}^{k} a_i\chi_{A_i}=\sum_{j=1}^{\ell} b_j\chi_{B_j} \Rightarrow \sum_i a_i\mu(A_i)=\sum_j b_j\mu(B_j)$). <u>Exercise</u>!

3) If $s_1,s_2: X\to[0,\infty)$ are simple, then $\int s_1+s_2=\int s_1+\int s_2$. <u>Exercise</u>!

4) If $s: X\to[0,\infty)$ is simple, $c\geq 0$, then $\int c\cdot s=c\cdot\int s$. — " —

5) If s_1,s_2 are simple, and $0\leq s_1\leq s_2$, then $\int s_1\leq\int s_2$.

Pf. The function $s_3:=s_2-s_1$ is simple and non-negative, hence

$$\int s_2=\int s_1+s_3 \underset{\text{by 3)}}{=} \int s_1+\int s_3\geq\int s_1,$$ b/c $\int s_3$ is the sum of non-negatives. □

<u>Def</u>. If $f: X\to\overline{\mathbb{R}}$ is measurable and $A\in\mathcal{M}$, we define the <u>integral of f over A</u> as

$$\int_A f\,d\mu := \int f\cdot\chi_A\,d\mu.$$

If $\int|f|\,d\mu<\infty$ (resp. $\int_A|f|\,d\mu<\infty$), we say that f is <u>integrable</u> (resp. over A).

<u>Prop</u>. Let $(X,\mathcal{M},\mu)$ be a measure space, $A,B\in\mathcal{M}$ with $B\subset A$, and $f,g: X\to\overline{\mathbb{R}}$ measurable.

Then:
1) $\int_B f=\int_A f\cdot\chi_B$, and f is integrable over B iff $f\cdot\chi_B$ is integrable over A.
2) If $f\leq g$, then $\int_A f\leq\int_A g$.
3) If $f\geq 0$, then $\int_B f\leq\int_A f$.
4) If $\mu(A)=0$, then $\int_A f=0$.
5) $(\inf f(A))\cdot\mu(A)\leq\int_A f\,d\mu\leq(\sup f(A))\cdot\mu(A)$.
6) If $f\geq 0$ and $\int_A f\,d\mu=0$, then $\mu(\{x\in A: f(x)\neq 0\})=0$. / We say: $f=0$ a.e. on A /
7) If $c\in\mathbb{R}^*$, then $\int_A c\cdot f=c\cdot\int_A f$, and f is integrable iff cf is so.
8) If f is integrable, then $\left|\int f\,d\mu\right|\leq\int|f|\,d\mu$.

Pf. Suppose first that $f,g\geq 0$.

1) follows from def'n.

2): sup over a bigger set is bigger.

3): $\int_B f \underset{\text{by 1)}}{=} \int_A f\cdot\chi_B \underset{\text{by 2)}}{\leq} \int_A f$. ✓

4) $\mu(A)=0 \Rightarrow \int_A s=0$ for every simple $0\leq s\leq f$ $\Rightarrow$ sup $=0$. ✓

5) The constant functions $\inf f(A)$ & $\sup f(A)$ satisfy: $\inf f(A) \le f \le \sup f(A)$, hence the result, by 2). ✓

6) Set, $\forall k \in \mathbb{Z}_+$, $B_k := \{x \in A \mid f(x) \ge \frac{1}{k}\}$. Then, $B_k \in \mathcal{M}$, $\forall k$, b/c f is measurable, and $B := \{x \in A \mid f(x) > 0\}$ satisfies $B = \bigcup_{k=1}^{\infty} B_k$.

Suppose $\mu(B) > 0$. Then $\exists k_0$ s.t. $\mu(B_{k_0}) > 0$. The simple function $s := \frac{1}{k_0} \cdot \chi_{B_{k_0}}$ then satisfies $0 \le s \le f$, and $\int_A s \, d\mu = \frac{1}{k_0} \cdot \mu(B_{k_0}) > 0$, hence $\int_A f \, d\mu > 0$, by 2). $\lightning$ ✓

7) The equality holds for simple functions, and s is a simple f'n satisfying $0 \le s \le f$ iff $c \cdot s$ is a simple f'n satisfying $0 \le c \cdot s \le c \cdot f$ (for $c > 0$). ✓

8) For $f \ge 0$, there is nothing to show. In general, knowing 2) for functions of arbitrary sign, we have $-|f| \le f \le |f|$, hence $-\int |f| = \int -|f| \le \int f \le \int |f|$, and thus $\left| \int f \right| \le \int |f|$. ✓

Now, for f, g of arbitrary sign, write $f = f^+ - f^-$, $g = g^+ - g^-$ and apply the above to the functions f^+, f^-, g^+, g^-.

Exercise: Fill in the details. □

Limit Theorems

Thm. (Monotone Convergence Thm) Suppose $f_n : X \to [0, \infty]$ are measurable and pointwise increasing. If $f = \lim_{n \to \infty} f_n$, then $\int f_n \, d\mu \xrightarrow[n \to \infty]{} \int f \, d\mu$.

Pf. By part 2) in the above proposition, the sequence $\left(\int f_n \, d\mu \right)_{n=1}^{\infty}$ is increasing. Moreover, the function $f = \lim_{n \to \infty} f_n$ is measurable, and so $\int f \, d\mu$ exists. Set $L := \lim_{n \to \infty} \int f_n \, d\mu$. Then, by part 2) in the above proposition, $\forall n$: $\int f_n \, d\mu \le \int f \, d\mu$, and so $L \le \int f \, d\mu$. We thus need to show that $L \ge \int f \, d\mu$.

Let $s = \sum_{i=1}^{k} a_i E_i$ be an arbitrary simple function with $a_i \ge 0$ and $0 \le s \le f$, and let $c \in (0,1)$ be arbitrary. For each n, set $A_n = \{x \in X : f_n(x) \ge c \cdot s(x)\}$. Then, $A_n \uparrow X$, since $f_n \nearrow f$ ptwise. Now, $\forall n$, $\int f_n \, d\mu \ge \int_{A_n} f_n \, d\mu \ge \int_{A_n} c \cdot s \, d\mu = c \cdot \int s \cdot \chi_{A_n} d\mu = c \cdot \sum_{i=1}^{k} a_i \mu(E_i \cap A_n)$

Letting $n \to \infty$, we get $L \ge c \cdot \sum_{i=1}^{k} a_i \mu(E_i) = c \cdot \int s \, d\mu$. Since c was arbitrary, $L \ge \int s \, d\mu$;

and since s was arbitrary, it follows that $L \geq \sup\{\int s : 0 \leq s \leq f\} = \int f$. □

Examples: Both hypotheses (≥ 0 & increasing) are necessary.

1) $X := [0,\infty)$, $f_n := -\frac{1}{n}$. Then $\int f_n \, dm = -\infty$, $\forall n$, but $f_n \nearrow 0$ and $\int 0 \, dm = 0$.

2) $X := \mathbb{R}$, $f_n := n \cdot \chi_{[0,\frac{1}{n}]}$. Then $\int f_n \, dm = n \cdot \frac{1}{n} = 1$, $\forall n$, but $f_n \to 0$ ptwise.

<u>Thm.</u> If $f, g : X \to \overline{\mathbb{R}}$ are both non-negative and measurable or both integrable, then so is $f+g$, and $\int f+g \, d\mu = \int f \, d\mu + \int g \, d\mu$.

Pf. The statement is true if f, g are ≥ 0 and simple.

Suppose then that f, g are ≥ 0 and measurable. Then $\exists (s_n), (t_n)$ sequences of simple functions, st. $0 \leq s_n \leq f$, $0 \leq t_n \leq g$, $\forall n$, and $s_n \nearrow f$, $t_n \nearrow g$. Then, $s_n + t_n \nearrow f+g$, and by the Monotone Conv. Thm.,
$$\int f+g \, d\mu = \lim_{n\to\infty} \int s_n + t_n \, d\mu = \lim_{n\to\infty} \left(\int s_n \, d\mu + \int t_n \, d\mu \right) =$$
$$= \lim_n \int s_n \, d\mu + \lim_n \int t_n \, d\mu = \int f \, d\mu + \int g \, d\mu.$$

Finally, if f, g are integrable, then so is $f+g$, b/c
$$\int |f+g| \leq \int (|f|+|g|) \underset{\text{by above}}{=} \int |f| + \int |g| < \infty.$$

Now, write
$$(f+g)^+ - (f+g)^- = f+g = f^+ - f^- + g^+ - g^-, \quad \text{hence}$$
$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$. Using the result for non-negative measurables, we get
$$\int (f+g)^+ + \int f^- + \int g^- = \int (f+g)^- + \int f^+ + \int g^+, \quad \text{hence}$$
$$\int (f+g)^+ - \int (f+g)^- = \left(\int f^+ - \int f^- \right) + \left(\int g^+ - \int g^- \right), \text{ as required. } \square$$

<u>Corollary</u>. If $f_n : X \to [0,\infty]$ are measurable, then $\int \left(\sum_{n=1}^{\infty} f_n \right) = \sum_{n=1}^{\infty} \int f_n$.

Pf. This is an equivalent statement of M.C.T. (applied to the sequence of partial sums). □

<u>Lemma</u> (Fatou's Lemma) If $f_n : X \to [0,\infty]$ are measurable, then
$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

Pf. Set $g_n := \inf_{k \geq n} f_k$. Then, $\forall n$, $0 \leq g_n \leq f_n$, and the sequence $(g_n(x))_{n=1}^{\infty}$ is increasing, $\forall x$.

Since, $\forall n \, \forall k \geq n$, $g_n \leq f_k$, we have $\int g_n \leq \int f_k$, $\forall k \geq n$, and hence $\int g_n \leq \inf_{k \geq n} \int f_k$.

Now, $\int \liminf_{n\to\infty} f_n = \int \sup_{n\geq 1}\left(\inf_{n\geq k} f_k\right) = \int \lim_{n\to\infty} g_n \overset{\text{M.C.T.}}{=} \lim_{n\to\infty}\int g_n \overset{g_n\leq g_{n+1}}{=} \sup_{n\geq 1}\int g_n \leq \sup_{n\geq 1}\inf_{k\geq n}\int f_k = \liminf_{n\to\infty}\int f_n$. □

Thm. (<u>Dominated Convergence Thm.</u>) Suppose $f_n : X \to \overline{\mathbb{R}}$ are measurable, and $g: X\to[0,\infty]$ is an integrable function s.t. $|f_n(x)|\leq g(x)$, $\forall x\in X$. Then:

(1) The functions f_n, $\forall n$, $\liminf_{n\to\infty} f_n$, $\limsup_{n\to\infty} f_n$ are integrable

(2) $\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty}\int f_n \leq \limsup_{n\to\infty}\int f_n \leq \int \limsup_{n\to\infty} f_n$.

In particular, if $f_n \xrightarrow[n\to\infty]{} f$ ptwise, then $\int f = \lim_{n\to\infty}\int f_n$.

Pf. (1) For all n, $|f_n|\leq g$ (by assumption), hence f_n are integrable.

Moreover, $-g(x)\leq f_n(x)\leq g(x)$, $\forall n, \forall x \Rightarrow |\liminf_{n\to\infty} f_n|\leq g$ and $|\limsup_{n\to\infty} f_n|\leq g$, and the two f'ns are measurable, hence integrable.

(2) For all n, $-g\leq f_n\leq g$ implies $g-f_n\geq 0$ and $f_n+g\geq 0$.

By Fatou's Lemma, $\int \liminf_{n\to\infty}(f_n+g) \leq \liminf_{n\to\infty}\int(f_n+g)$, hence

$$\int g+\int\liminf_{n\to\infty} f_n = \int\left(g+\liminf_{n\to\infty} f_n\right) \qquad \liminf_{n\to\infty}\left(\int f_n+\int g\right) = \liminf_{n\to\infty}\int f_n+\int g.$$

Similarly, $\int g + \int \liminf_{n\to\infty}(-f_n) \leq \int g + \liminf_{n\to\infty}\int(-f_n)$, and since $\liminf_{n\to\infty}(-f_n) = -\limsup_{n\to\infty} f_n$, we get

$-\int\limsup_{n\to\infty} f_n \leq -\limsup_{n\to\infty}\int f_n$, or $\int\limsup_{n\to\infty} f_n \geq \limsup_{n\to\infty}\int f_n$ □

<u>VIII</u>. <u>Comparison of Riemann & Lebesgue Integrals</u>.

<u>Def.</u> Let (X,d) be a metric space, and let $f: X\to\overline{\mathbb{R}}$. Define $m_f, M_f : X\to\overline{\mathbb{R}}$ by

$M_f(x) := \inf\{\sup f(U) : U \text{ open nbhd of } x\}$

$m_f(x) := \sup\{\inf f(U) : U \text{ open nbhd of } x\}$, $x\in X$.

<u>Remarks</u>:

1) $m_f\leq f\leq M_f$, for any f.

Exercise!

2) The functions m_f, M_f are Borel measurable, since $\forall c\in\mathbb{R}$, $\{x\in X: M_f(x)<c\}$ are $\{x\in X : m_f(x)>c\}$ open.

3) f is continuous at $x_0 \iff m_f(x_0) = M_f(x_0)$.

Notation: Given $[a,b] \subset \mathbb{R}$ and a bounded function $f:[a,b] \to \mathbb{R}$, let

$$S(f) := \Big\{ \lim_{n\to\infty} S(f, P^{(n)}, \xi^{(n)}) : \begin{array}{l} (P^{(n)})_{n=1}^{\infty} \text{ is a nested sequence of partitions of } [a,b], \\ (\xi^{(n)})_{n=1}^{\infty} \text{ sequence of sample points} \\ (S(f,P^{(n)},\xi^{(n)}))_{n=1}^{\infty} \text{ convergent seq. of Riemann sums} \end{array} \Big\}$$

We will denote the intervals of $P^{(n)}$ by $I_1^{(n)}, \dots, I_{\nu(n)}^{(n)}$, and $\xi^{(n)} = \{\xi_1^{(n)}, \dots, \xi_{\nu(n)}^{(n)}\}$ with $\xi_i^{(n)} \in I_i^{(n)}$.

<u>Lemma.</u> Let $I = [a,b]$ and let $f: I \to \mathbb{R}$ be a bounded function.

Let $m(f) = \int_I m_f \, dm$ and $M(f) = \int_I M_f \, dm$. Then,

$$\{m(f), M(f)\} \subset S(f) \subset [m(f), M(f)].$$

Pf. We will only prove that $M(f) \in S(f) \subset (-\infty, M(f)]$. The proof for $m(f)$ is analog's.

Let $\alpha \in S(f)$ be arbitrary, and let $(P^{(n)})_{n=1}^{\infty}$, $(\xi^{(n)})_{n=1}^{\infty}$ be s.t. $\lim_{n\to\infty} S(f, P^{(n)}, \xi^{(n)}) = \alpha$.

Define, for $n \in \mathbb{Z}_+$,

$$s_n := \sum_{i=1}^{\nu(n)} f(\xi_i^{(n)}) \cdot \chi_{\mathrm{int}(I_i^{(n)})}$$

$$\bar{s}_n := \sum_{i=1}^{\nu(n)} (\sup f(I_i^{(n)})) \cdot \chi_{\mathrm{int}(I_i^{(n)})}$$

$$\tilde{s}_n := \sum_{i=1}^{\nu(n)} f(\eta_i^{(n)}) \cdot \chi_{\mathrm{int}(I_i^{(n)})}$$

, where $\eta^{(n)} = \{\eta_1^{(n)}, \dots, \eta_{\nu(n)}^{(n)}\}$ are sample pts for $P^{(n)}$ s.t. $f(\eta_i^{(n)}) \geq \max\{f(\xi_i^{(n)}), \sup f(I_i^{(n)}) - \frac{1}{n}\}$, $\forall i$.

Then, the $s_n, \bar{s}_n, \tilde{s}_n$ are simple functions,

$\int_I s_n = S(f, P^{(n)}, \xi^{(n)})$, and $s_n \leq \tilde{s}_n \leq \bar{s}_n$, $\forall n$. Moreover, $0 \leq \bar{s}_n - \tilde{s}_n \leq \frac{1}{n}$, $\forall n$.

Set $A := \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\nu(n)} \mathrm{int}(I_i^{(n)})$. Then, $A \in \mathcal{B}(I)$ and $m(I \setminus A) = 0$.

Observe that, $\forall x \in A$, $\bar{s}_n(x) \xrightarrow[n\to\infty]{} M_f(x)$, and hence also $\tilde{s}_n(x) \xrightarrow[n\to\infty]{} M_f(x)$.

Now,

$$S(f, P^{(n)}, \xi^{(n)}) = \int_I s_n \, dm \underset{m(I\setminus A)=0}{=} \int_A s_n \, dm \leq \int_A \bar{s}_n \, dm \xrightarrow[n\to\infty]{} \int_A M_f \, dm = \int_I M_f \, dm = M(f).$$

by Dominated Conv. Thm., b/c $|f|$ bdd by $C \Rightarrow |s_n|, |\tilde{s}_n|, |\bar{s}_n|$ bdd by C, $\forall n$

Hence, $\alpha \leq M(f)$, and so $S(f) \subset (-\infty, M(f)]$.

On the other hand,

$$S(f, P^{(n)}, \eta^{(n)}) = \int_I \tilde{s}_n \, dm = \int_A \tilde{s}_n \, dm \xrightarrow[n\to\infty]{\text{Dom. Conv. Thm.}} \int_A M_f \, dm = \int_I M_f \, dm = M(f), \text{ so } M(f) \in S(f). \ \square$$

<u>Thm.</u> Let $a < b$, $I = [a,b]$, $f: I \to \mathbb{R}$ bounded, and let $N(f) := \{x \in I : f \text{ not cont's at } x\}$.

Then: (1) f is Riemann integrable iff $m(N(f)) = 0$.

(2) If the equiv. conditions of (1) are satisfied, then f is Lebesgue integrable and

$$\int_I f \, dm = \int_a^b f(x) \, dx.$$

Pf. (1): We have

$$f \text{ Riemann integrable} \iff |S(f)| = 1 \iff m(f) = M(f) \iff \int_I (M_f - m_f)\, dm = 0 \iff$$

$$\iff m(\{x \in I : M_f(x) > m_f(x)\}) = 0, \text{ but } \{x \in I : M_f(x) > m_f(x)\} = N(f). \checkmark$$

(2) Suppose $m(N(f)) = 0$.

Note that $N(f) \in \mathcal{B}(I)$, since the functions m_f and M_f are Borel measurable.

We have $f|_{I \setminus N(f)} = m_f|_{I \setminus N(f)}$, so $f = m_f$ a.e., and hence f is Lebesgue measurable (Exercise!)

Finally, $\int_I f\, dm = \int_{I \setminus N(f)} f\, dm = \int_{I \setminus N(f)} m_f\, dm = \int_I m_f\, dm = m(f) \in S(f)$, and the latter is the singleton $\int_a^b f(x)\, dx$. $\square$

IX. Integration on Product Spaces

Def. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. A measurable rectangle is any set of the form $A \times B$, where $A \in \mathcal{M}$, $B \in \mathcal{N}$. The algebra of finite disjoint unions of measurable rectangles, denoted $\mathcal{E}$, is called the algebra of elementary sets. The product σ-algebra of $\mathcal{M}$ and $\mathcal{N}$ is $\mathcal{M} \times \mathcal{N} := \sigma(\mathcal{E})$.

Given $E \subset X \times Y$, $x_0 \in X$, $y_0 \in Y$, the x_0-section and y_0-section of E are defined as

$$E_{x_0} = \{y \in Y : (x_0, y) \in E\}, \quad E^{y_0} := \{x \in X : (x, y_0) \in E\}.$$

Prop. If $E \in \mathcal{M} \times \mathcal{N}$, then $E_x \in \mathcal{N}$ and $E^y \in \mathcal{M}$, for all $x \in X$, $y \in Y$.

Pf. We shall prove the result for E_x. The proof for E^y is analogous.

Let $\Omega := \{E \in \mathcal{M} \times \mathcal{N} : E_x \in \mathcal{N}, \forall x \in X\}$.

If $E = A \times B$ for some $A \in \mathcal{M}$, $B \in \mathcal{N}$, then $E_x = \begin{cases} B, & x \in A \\ \emptyset, & x \in X \setminus A \end{cases}$, and so Ω contains all measurable rectangles. Moreover,

(i) $X \times Y \in \Omega$ (as a m'ble rectangle)

(ii) $E \in \Omega \Rightarrow E^c \in \Omega$, b/c $(E^c)_x = \{y : (x,y) \in E^c\} = \{y : (x,y) \in E\}^c = (E_x)^c \in \mathcal{N}$, $\forall x$.

(iii) $\{E_n\}_{n=1}^{\infty} \subset \Omega \Rightarrow \bigcup_{n=1}^{\infty} E_n \in \Omega$, b/c $(\bigcup_n E_n)_x = \bigcup_n (E_n)_x \in \mathcal{N}$, $\forall x$.

Thus, Ω is a σ-algebra containing $\mathcal{E}$, and so $\Omega \supset \mathcal{M} \times \mathcal{N}$, by minimality. $\square$

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Def. For a function $f: X\times Y \to \overline{\mathbb{R}}$ and points $x_0\in X$, $y_0\in Y$, we define the x_0-section and y_0-section of f as $f_{x_0}: Y \ni y \mapsto f(x_0,y)\in\overline{\mathbb{R}}$, $f^{y_0}: X\ni x\mapsto f(x,y_0)\in\overline{\mathbb{R}}$.

Prop. Let $(X,\mathcal{M})$, $(Y,\mathcal{N})$ be measurable spaces, and let $f: X\times Y\to\overline{\mathbb{R}}$.

If f is $\mathcal{M}\times\mathcal{N}$-measurable, then:

(i) f_x is $\mathcal{N}$-measurable, for all $x\in X$.

(ii) f^y is $\mathcal{M}$-measurable, for all $y\in Y$.

Pf. Given $x_0\in X$, let $U\subset\overline{\mathbb{R}}$ be an arbitrary open set. Set $E := f^{-1}(U) = \{(x,y)\in X\times Y: f(x,y)\in U\}$. Then, $E\in\mathcal{M}\times\mathcal{N}$, by assumption, and hence $E_{x_0}\in\mathcal{N}$, by the previous proposition. But, $E_{x_0} = \{(x_0,y)\in X\times Y: f(x_0,y)\in U\} = (f_{x_0})^{-1}(U)$. This proves that f_{x_0} is $\mathcal{N}$-measurable, since U was arbitrary. □

Thm. Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be σ-finite measure spaces. Let $Q\in\mathcal{M}\times\mathcal{N}$, and let $\varphi: X\to\overline{\mathbb{R}}$ and $\psi: Y\to\overline{\mathbb{R}}$ be defined as $\varphi(x) := \nu(Q_x)$, $\psi(y) := \mu(Q^y)$. Then, φ is $\mathcal{M}$-measurable, ψ is $\mathcal{N}$-measurable, and $\int_X \varphi(x)\,d\mu(x) = \int_Y \psi(y)\,d\nu(y)$.

Pf. Let $\Omega\subset\mathcal{M}\times\mathcal{N}$ be the collection of all sets for which the conclusion of the thm. holds. We will show that Ω satisfies the following:

(a) $A\in\mathcal{M}$, $B\in\mathcal{N}$ $\Rightarrow$ $A\times B\in\Omega$

(b) $\{Q_n\}_{n=1}^\infty\subset\Omega$, $Q_n\uparrow Q$ $\Rightarrow$ $Q\in\Omega$

(c) $\{Q_n\}_{n=1}^\infty\subset\Omega$, $Q_i\cap Q_j=\emptyset$ $\forall i\neq j$ $\Rightarrow$ $\bigcup_{n=1}^\infty Q_n\in\Omega$.

(d) If $A\in\mathcal{M}$, $B\in\mathcal{N}$, $\mu(A)<\infty$, $\nu(B)<\infty$, and $A\times B\supset Q_1\supset Q_2\supset Q_3\supset\dots$ with $\{Q_n\}\subset\Omega$, then $\bigcap_{n=1}^\infty Q_n\in\Omega$.

For the proof of (a), let $A\in\mathcal{M}$, $B\in\mathcal{N}$, and let $Q = A\times B$.

Then, $\forall x\in X$ $\forall y\in Y$, $\nu(Q_x) = \nu(B)\cdot\chi_A(x)$ and $\mu(Q^y) = \mu(A)\cdot\chi_B(y)$, so φ and ψ are measurable as simple functions, and $\int_X \varphi(x)\,d\mu(x) = \int_X \nu(B)\cdot\chi_A(x)\,d\mu(x) = \nu(B)\cdot\mu(A) = \int_Y \mu(A)\cdot\chi_B\,d\nu(y) = \int_Y \psi(x)\,d\nu(y)$. ✓

For (b), let $\varphi_i(x) = \nu((Q_i)_x)$, $\psi_i(x) = \mu((Q_i)^y)$, for $i\in\mathbb{Z}_+$. Since $Q_n\uparrow Q$, it follows that $(\forall x,\forall y)$ $(Q_n)_x\uparrow Q_x$ and $(Q_n)^y\uparrow Q^y$, hence $\varphi_n(x)\nearrow\varphi(x)$ and $\psi_n(y)\nearrow\psi(y)$, $\forall x$ $\forall y$.

By assumption, $\forall n$, $\int_X \varphi_n(x)\, d\mu(x) = \int_Y \psi_n(y)\, d\nu(y)$, and by the Monotone Conv. Thm.,
$\int_X \varphi_n\, d\mu \xrightarrow[n\to\infty]{} \int_X \varphi\, d\mu$ and $\int_Y \psi_n\, d\nu \xrightarrow[n\to\infty]{} \int_Y \psi\, d\nu$. Thus, $\int_X \varphi\, d\mu = \int_Y \psi\, d\nu$, as required. ✓

For (c), let $\varphi_i(x) = \nu((Q_1 \sqcup \dots \sqcup Q_i)_x)$ and $\psi_i(y) = \mu((Q_1 \sqcup \dots \sqcup Q_i)^y)$, $i \in \mathbb{Z}_+$. Let $Q := \bigsqcup_{n=1}^{\infty} Q_n$.

Then, $\forall n$, $\int_X \varphi_n(x)\, d\mu(x) = \int_X \nu((Q_1)_x \sqcup \dots \sqcup (Q_n)_x)\, d\mu(x) = \int_X \sum_{i=1}^{n} \nu((Q_i)_x)\, d\mu(x) = \sum_{i=1}^{n} \int_X \nu((Q_i)_x)\, d\mu(x) =$

$$= \sum_{i=1}^{n} \int_Y \mu((Q_i)^y)\, d\nu(y) = \dots = \int_Y \psi_n(y)\, d\nu(y),$$

hence $Q_1 \sqcup \dots \sqcup Q_n \in \Omega$, $\forall n$. The claim now follows from part (b). ✓

For (d), let $\varphi_i(x) = \nu((Q_i)_x)$ and $\psi_i(y) = \mu((Q_i)^y)$, $i \in \mathbb{Z}_+$. Let $Q := \bigcap_{n=1}^{\infty} Q_n$.

Then, as in the proof of (b), $\varphi_n(x) \searrow \varphi(x)$ and $\psi_n(y) \searrow \psi(y)$, $\forall x\ \forall y$. Moreover, φ_1 and ψ_1 (and hence all φ_n, ψ_n) are bounded by an integrable function. Indeed,

$\varphi_1(x) = \nu((Q_1)_x) \leq \nu((A\times B)_x) = \nu(B)\cdot \chi_A(x)$, $\forall x$, so $\varphi_1 \leq \nu(B)\cdot\chi_A$ (and $\int_X \nu(B)\cdot\chi_A = \nu(B)\cdot\mu(A) < \infty$),

and $\psi_1(y) = \mu((Q_1)^y) \leq \mu((A\times B)^y) = \mu(A)\cdot\chi_B(y)$, $\forall y$, so $\psi_1 \leq \mu(A)\cdot\chi_B$.

Thus, by the Dominated Conv. Thm., $\int_X \varphi_n\, d\mu \xrightarrow[n\to\infty]{} \int_X \varphi\, d\mu$ and $\int_Y \psi_n\, d\nu \xrightarrow[n\to\infty]{} \int_Y \psi\, d\nu$.
Since $\int_X \varphi_n\, d\mu = \int_Y \psi_n\, d\nu$, $\forall n$, the claim follows. ✓

Now, let $\{X_n\}_{n=1}^{\infty} \subset \mathcal{M}$ and $\{Y_n\}_{n=1}^{\infty} \subset \mathcal{N}$ be families of pairwise disjoint measurable sets s.t.
$X = \bigcup_n X_n$, $Y = \bigcup_n Y_n$, $\mu(X_n) < \infty$, $\forall n$, and $\nu(Y_n) < \infty$, $\forall n$.

For every $Q \subset X \times Y$, set $Q_{mn} := Q \cap (X_m \times Y_n)$, for $m,n \in \mathbb{Z}_+$.

Define $\mathcal{C} := \{Q \in \mathcal{M}\times\mathcal{N} : Q_{mn} \in \Omega \text{ for all } m,n \in \mathbb{Z}_+\}$.

Then, by (b) and (d) above, $\mathcal{C}$ is a monotone class, and by (a) and (c), $\mathcal{C}$ contains all the elementary sets $\mathcal{E}$. Since by the Monotone Class Thm., $\mathcal{M}\times\mathcal{N}$ is the monotone class generated by $\mathcal{E}$, it follows that $\mathcal{C} \supset \mathcal{M}\times\mathcal{N}$; hence, $\mathcal{C} = \mathcal{M}\times\mathcal{N}$.

Therefore, $\forall Q \in \mathcal{M}\times\mathcal{N}$, we have $Q_{mn} \in \Omega$, $\forall m,n \in \mathbb{Z}_+$. But $Q = \bigsqcup_{m,n=1}^{\infty} Q_{mn}$ and the elt's of this union are pairwise disjoint, so $Q \in \Omega$, by (c). $\square$

<u>Corollary + Def.</u> Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. There exists a unique σ-finite measure on $\mathcal{M}\times\mathcal{N}$, denoted $\mu\times\nu$, satisfying $(\mu\times\nu)(A\times B) = \mu(A)\cdot\nu(B)$ for all $A\in\mathcal{M}$, $B\in\mathcal{N}$. Moreover, $\forall E\in\mathcal{M}\times\mathcal{N}$,

$$(\mu\times\nu)(E) = \int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E^y)\, d\nu(y).$$

The $\mu\times\nu$ is called the <u>product measure</u> on $\mathcal{M}\times\mathcal{N}$.

Pf. That $\mu\times\nu$ defined by the above equality is a measure on $\mathcal{M}\times\mathcal{N}$ follows from the proof of (c) in the above theorem. Uniqueness of $\mu\times\nu$ follows from proof 4) in the Carathéodory Extension Thm., since $\mu\times\nu$ is a premeasure on $\mathcal{E}$. $\square$

<u>Thm. (Fubini, I)</u> Let $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and let $f: X\times Y \to \overline{\mathbb{R}}$ be $\mathcal{M}\times\mathcal{N}$-measurable. Suppose $f\geq 0$ or f is integrable.

Then:

(1) The functions $X\ni x\mapsto \int_Y f(x,y)\, d\nu(y)\in[0,\infty]$

$Y\ni y\mapsto \int_X f(x,y)\, d\mu(x)\in[0,\infty]$ are $\mathcal{M}$- and $\mathcal{N}$-measurable, resp.

(2) We have

$$\int_{X\times Y} f\, d(\mu\times\nu) = \int_X\left(\int_Y f(x,y)\, d\nu(y)\right) d\mu(x) = \int_Y\left(\int_X f(x,y)\, d\mu(x)\right) d\nu(y).$$

Pf.

Suppose first that $f = \chi_E$ for some $E\in\mathcal{M}\times\mathcal{N}$. Then, $\forall x\in X$, $f(x,y) = \chi_E(x,y) = \chi_{E_x}(y)$, and hence $\int_Y f(x,y)\, d\nu(y) = \int_Y \chi_{E_x}\, d\nu = \nu(E_x)$, so that the first function in (1) is the function φ from the previous theorem, hence measurable. Similarly, the second function in (1) is the ψ from the previous thm., hence measurable. By that thm. as well, $\int_X \varphi\, d\mu = \int_Y \psi\, d\mu$ (by above Cor.), which proves (2).

Suppose next that f is a non-negative simple function. We can write $f = \sum_{i=1}^n a_i \chi_{E_i}$, for some $a_i\geq 0$ and pairwise disjoint $E_1,\dots,E_n\in\mathcal{M}\times\mathcal{N}$. The claim then follows from the above case and linearity of the integral.

Finally, suppose f is an arbitrary non-negative $\mathcal{M}\times\mathcal{N}$-measurable function. Let $(s_n)_{n=1}^\infty$ be a sequence of simple functions s.t. $0\leq s_n\leq f$ and $s_n\nearrow f$ $(n\to\infty)$.

Then, $\forall x\ \forall y$, $((s_n)_x)_{n=1}^\infty$ (resp. $((s_n)^y)_{n=1}^\infty$) is a seq. of simple f'ns with $0\leq(s_n)_x\leq f_x$ (resp. $0\leq(s_n)^y\leq f^y$) and $(s_n)_x\nearrow f_x$ (resp. $(s_n)^y\nearrow f^y$). Thus, by the Monotone Conv. Thm., $\forall x\in X$, $\int_Y f_x\, d\nu = \lim_{n\to\infty}\int_Y (s_n)_x\, d\nu$, and so the f'n $\{X\ni x\mapsto\int_Y f_x\, d\nu\in\overline{\mathbb{R}}\}$ is the limit of f's $\{X \ni x \mapsto \int_Y (s_n)_x\, d\nu \in \mathbb{R}\}$, which are $\mathcal{M}$-measurable, hence it is $\mathcal{M}$-measurable. Similarly, $\{Y \ni y \mapsto \int_X f^y\, d\mu\}$ is the limit of f's $\{Y \ni y \mapsto \int_X (s_n)^y\, d\mu \in \mathbb{R}\}$, and hence is $\mathcal{N}$-measurable. This proves (1).

Part (2) follows from the equalities $\int_X \left(\int_Y (s_n)_x\, d\nu\right) d\mu = \int_Y \left(\int_X (s_n)^y\, d\mu\right) d\nu = \int_{X\times Y} s_n\, d(\mu\times\nu)$, $\forall n$, and the Monotone Conv. Thm., as follows:

$$\int_X \left(\int_Y f_x\, d\nu\right) d\mu = \int_X \left(\lim_{n\to\infty} \int_Y (s_n)_x\, d\nu\right) d\mu \underset{\substack{\uparrow \\ \int_Y (s_n)_x d\nu \le \int_Y (s_{n+1})_x d\nu,\ \text{since } (s_n)_x \le (s_{n+1})_x}}{=} \lim_{n\to\infty} \int_X \left(\int_Y (s_n)_x\, d\nu\right) d\mu = \lim_{n\to\infty} \int_{X\times Y} s_n\, d(\mu\times\nu) \underset{\substack{\uparrow \\ s_n \nearrow f}}{=} \int_{X\times Y} f\, d(\mu\times\nu) =$$

$$= \lim_{n\to\infty} \int_Y \left(\int_X (s_n)^y\, d\mu\right) d\nu = \int_Y \left(\lim_{n\to\infty} \int_X (s_n)^y\, d\mu\right) d\nu = \int_Y \left(\int_X f^y\, d\mu\right) d\nu.$$

This completes the proof for $f \geq 0$.

If f is integrable, then $\int_{X\times Y} f^+$ and $\int_{X\times Y} f^-$ are $<\infty$ and hence $\int_X \left(\int_Y f_x^{+/-}\, d\nu\right) d\mu < \infty$ and $\int_Y \left(\int_X (f^{+/-})^y\, d\mu\right) d\nu < \infty$, so we can repeat the above case for f^+ and f^- and get formula (2) for f from the corresponding formulas for f^+ and f^-. $\square$

Completion of Product Measures

Even if $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are complete measure spaces, it may happen that $(X\times Y, \mathcal{M}\times\mathcal{N}, \mu\times\nu)$ is not complete. E.g., if $A \in \mathcal{M}$, $A \neq \emptyset$ and $\mu(A) = 0$, and $B \in \mathcal{P}(Y) \setminus \mathcal{N}$, then $E := A \times B \notin \mathcal{M}\times\mathcal{N}$ (b/c $\forall x \in A$, $E_x = B$ is non-measurable), but $A \times B \subset A \times Y$ and $(\mu\times\nu)(A\times Y) = \mu(A)\cdot\nu(Y) = 0\cdot\nu(Y) = 0$. This happens, for instance, in case of Lebesgue measures. Nonetheless, we have:

Thm. Let $n \geq 2$, and let $k, l \in \mathbb{Z}_+$ be s.t. $n = k + l$. Let $\mathcal{B}_i$, $\mathcal{L}_i$ denote the σ-algebras of Borel- and Lebesgue-measurable sets in $\mathbb{R}^i$ ($i \in \mathbb{Z}_+$), respectively. Then, $\mathcal{B}_n \subset \mathcal{L}_k \times \mathcal{L}_l \subset \mathcal{L}_n$, and m_n is the completion of $m_k \times m_l$.

Pf. Clearly, every open n-box $I^1 \times \ldots \times I^n$ belongs to $\mathcal{L}_k \times \mathcal{L}_l$ (since $I^1 \times \ldots \times I^n = (I^1 \times \ldots \times I^k) \times (I^{k+1} \times \ldots \times I^n)$). Since $\mathcal{B}_n$ is the σ-algebra generated by open n-boxes, we get $\mathcal{B}_n \subset \mathcal{L}_k \times \mathcal{L}_l$.

Next, let $E \times F$ be a (Lebesgue) measurable rectangle in $\mathbb{R}^k \times \mathbb{R}^l$. Then, $\exists A, B \in \mathcal{B}_k$ $\exists C, D \in \mathcal{B}_l$ s.t. $A \subset E \subset B$, $C \subset F \subset D$, $m_k(B \setminus A) = 0$, and $m_l(D \setminus C) = 0$. Now, $A \times \mathbb{R}^l$, $B \times \mathbb{R}^l$, $\mathbb{R}^k \times C$, $\mathbb{R}^k \times D$ are all in $\mathcal{B}_n$, and $A\times\mathbb{R}^\ell \subset E\times\mathbb{R}^\ell \subset B\times\mathbb{R}^\ell$, $\mathbb{R}^k\times C\subset\mathbb{R}^k\times F\subset\mathbb{R}^k\times D$. It is easy to see that $m_n((B\times\mathbb{R}^\ell)\setminus(A\times\mathbb{R}^\ell)) = 0 = m_n((\mathbb{R}^k\times D)\setminus(\mathbb{R}^k\times C))$, and hence $E\times\mathbb{R}^\ell, \mathbb{R}^k\times F\in\mathcal{L}_n$. (Indeed, given $\varepsilon>0$, let for every $i\in\mathbb{Z}_+$, $\{U_j^i\}_{j=1}^\infty$ be a collection of open k-boxes s.t. $B\setminus A\subset\bigcup_{j=1}^\infty U_j^i$ and $\sum_{j=1}^\infty m_k(U_j^i)\le\frac{\varepsilon}{2^{i+1}}$. Then $(B\times[0,\infty))\setminus(A\times[0,\infty)) = (B\setminus A)\times[0,\infty)\subseteq\bigcup_{i,j=1}^\infty(U_j^i\times[i-1,i))$ and $m_n\Big(\bigcup_{i,j}U_j^i\times[i-1,i)\Big)\le\sum_{i=1}^\infty\frac{\varepsilon}{2^{i+1}}=\frac{\varepsilon}{2}$. Thus, $m_n((B\setminus A)\times[0,\infty))\le\frac{\varepsilon}{2}$. Similarly, $m_n((B\setminus A)\times(-\infty,0])\le\frac{\varepsilon}{2}$, so $m_n((B\setminus A)\times\mathbb{R})\le\varepsilon$. This proves the claim for $\ell=1$, the general case follows by induction.)

Hence, $E\times F = (E\times\mathbb{R}^\ell)\cap(\mathbb{R}^k\times F)\in\mathcal{L}_n$. Since $\mathcal{L}_k\times\mathcal{L}_\ell$ is the σ-algebra generated by all the Lebesgue measurable rectangles in $\mathbb{R}^k\times\mathbb{R}^\ell$, it follows that $\mathcal{L}_k\times\mathcal{L}_\ell\subset\mathcal{L}_n$.

Let now $Q\in\mathcal{L}_k\times\mathcal{L}_\ell$ be arbitrary. By the above, $Q\in\mathcal{L}_n$, so we can choose $P,R\in\mathcal{B}_n$ such that $P\subset Q\subset R$ and $m_n(R\setminus P)=0$. Since both $m_k\times m_\ell$ and m_n assign the same values to n-boxes, it follows that they agree on Borel measurable sets (by σ-finiteness and Caratheodory Ext. Thm. (4)). Hence,

$$(m_k\times m_\ell)(Q\setminus P)\le(m_k\times m_\ell)(R\setminus P) = m_n(R\setminus P) = 0, \text{ and thus}$$

$$(m_k\times m_\ell)(Q) = (m_k\times m_\ell)(Q\setminus P) + (m_k\times m_\ell)(P) = (m_k\times m_\ell)(P) = m_n(P) = m_n(Q).$$

Therefore, $m_k\times m_\ell$ and m_n agree on $\mathcal{L}_k\times\mathcal{L}_n$. The result now follows from the uniqueness of completion! $\square$

<u>Thm. (Fubini, II)</u> Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be complete σ-finite measure spaces. Let $\overline{(\mathcal{M}\times\mathcal{N})}$ be the completion of $\mathcal{M}\times\mathcal{N}$, relative to the product measure $\mu\times\nu$. Let $f: X\times Y\to\mathbb{R}$ be an $\overline{(\mathcal{M}\times\mathcal{N})}$-measurable function. Suppose $f\ge0$ or integrable. Then:

(1) $\mu(\{x\in X: f_x \text{ is not } \mathcal{N}\text{-measurable}\})=0$ and $\nu(\{y\in Y: f^y \text{ is not } \mathcal{M}\text{-measurable}\})=0$.

(2) After setting to 0 over the above sets, the functions

$X\ni x\mapsto\int_Y f_x(y)\,d\nu(y)$ and $Y\ni y\mapsto\int_X f^y(x)\,d\mu(x)$ are $\mathcal{M}$- and $\mathcal{N}$-measurable, resp., and the integral formula from Fubini Thm. holds.

We first establish the following two lemmas.

<u>Lemma 1.</u> Let α be a measure on a σ-algebra $\mathcal{M}$, let $\overline{\mathcal{M}}$ be the completion of $\mathcal{M}$ rel. α, and let f be $\overline{\mathcal{M}}$-measurable. Then, $\exists$ $\mathcal{M}$-measurable g s.t. $f=g$ a.e. (rel. $\overline{\alpha}$).

Pf. Suppose f is $\overline{\mathcal{M}}$-measurable and $f \geq 0$. Let $(s_n)_{n=1}^{\infty}$ be a sequence of simple functions s.t. $0 \leq s_n \leq f$ and $s_n \nearrow f$ $(n \to \infty)$. Then, $f = \left(\sum_{n=1}^{\infty}(s_{n+1} - s_n)\right) + s_1$, and since each $s_{n+1} - s_n$ is a finite linear combination of characteristic functions, it follows that $f = \sum_{i=1}^{\infty} c_i \chi_{E_i}$ for some $\{E_i\}_{i=1}^{\infty} \subset \overline{\mathcal{M}}$, $c_i \geq 0$.

Now, by def'n of $\overline{\mathcal{M}}$, $\forall i \in \mathbb{Z}_+$, $\exists A_i, B_i \in \mathcal{M}$ s.t. $A_i \subset E_i \subset B_i$ and $\alpha(B_i \setminus A_i) = 0$. Define $g := \sum_{i=1}^{\infty} c_i \chi_{A_i}$. Then, g is $\mathcal{M}$-measurable and $f(x) = g(x)$, $\forall x \in \left(\bigcup_{i=1}^{\infty}(E_i \setminus A_i)\right)^c$. But $\bigcup_{i=1}^{\infty}(E_i \setminus A_i) \subset \bigcup_{i=1}^{\infty}(B_i \setminus A_i)$, so $\overline{\alpha}\left(\bigcup_{i=1}^{\infty}(E_i \setminus A_i)\right) = 0$, as required. (For arbitrary f, $f = f^+ - f^-$ and the result follows from above.) $\square$

Lemma 2. Let h be an $\overline{\mathcal{M} \times \mathcal{N}}$-measurable function on $X \times Y$ s.t. $h = 0$ a.e. rel. $\overline{\mu \times \nu}$. Then, $\exists A \in \mathcal{M}$, $B \in \mathcal{N}$ s.t. $\mu(A) = 0 = \nu(B)$ and $h_x = 0$ a.e. rel. ν, for all $x \in X \setminus A$, $h^y = 0$ a.e. rel. μ, for all $y \in Y \setminus B$.
In particular, h_x (resp. h^y) is $\mathcal{N}$- (resp. $\mathcal{M}$-) measurable, $\forall x \in A^c$ $\forall y \in B^c$.

Pf. Set $P := \{(x,y) \in X \times Y : h(x,y) \neq 0\}$. Then $P \in \overline{\mathcal{M} \times \mathcal{N}}$ and $(\overline{\mu \times \nu})(P) = 0$. Then, there is $Q \in \mathcal{M} \times \mathcal{N}$ s.t. $P \subset Q$ and $(\mu \times \nu)(Q) = 0$. By def'n of $\mu \times \nu$, it means that $\int_X \nu(Q_x)\, d\mu(x) = 0$. $(*)$
Set $A := \{x \in X : \nu(Q_x) > 0\}$. By the theorem preceding the def'n of $\mu \times \nu$, $\{x \mapsto \nu(Q_x)\}$ is $\mathcal{M}$-measurable, and hence $A \in \mathcal{M}$. By $(*)$, $\mu(A) = 0$.
Now, for every $x_0 \in X \setminus A$, we have $h_{x_0}(y) \neq 0$ only if $y \in P_{x_0}$. But $P_{x_0} \subset Q_{x_0}$, and since $\nu(Q_{x_0}) = 0$ and $(Y, \mathcal{N}, \nu)$ is complete, we have $P_{x_0} \in \mathcal{N}$ and $\nu(P_{x_0}) = 0$. Thus, $h_{x_0} = 0$ on $(P_{x_0})^c$, which is a.e. rel. ν. The proof for h^y is analogous. $\square$

Proof of Fubini II: Let f be $\overline{\mathcal{M} \times \mathcal{N}}$-measurable. By Lemma 1 (applied to $\overline{\alpha} = \overline{\mu \times \nu}$), we can write $f = g + h$ for some $\mathcal{M} \times \mathcal{N}$-measurable g, and $h = 0$ a.e. rel $\overline{\mu \times \nu}$. Now, the (classic) Fubini Thm. applies to g, and Lemma 2 shows that $f_x = g_x$, $\forall x \in X \setminus A$ and $f^y = g^y$, $\forall y \in Y \setminus B$ for some $A \in \mathcal{M}$, $B \in \mathcal{N}$ with $\mu(A) = 0 = \nu(B)$. $\square$