X. SIGNED MEASURES

Def. Let $\mathcal{M}$ be a σ-algebra. A signed measure on $\mathcal{M}$ is a function $\mu : \mathcal{M} \to (-\infty, \infty]$ such that: (i) $\mu(\emptyset) = 0$

(ii) $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$, $A_i \cap A_j = \emptyset \ \forall i \neq j \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$;

and if $\mu(\bigcup_n A_n) < \infty$ then the series converges absolutely.

Terminology: From now on we will refer to (ordinary) measures as positive measures.

Def. Let μ be a signed measure on a σ-algebra $\mathcal{M}$. A set $A \in \mathcal{M}$ is called a positive set (for μ) when $\mu(B) \geq 0$ for all $B \in \mathcal{M} \cap \mathcal{P}(A)$. A set $A \in \mathcal{M}$ is a negative set (for μ) when $\mu(B) \leq 0$ for all $B \in \mathcal{M} \cap \mathcal{P}(A)$.
A is a null set (for μ) when $\mu(B) = 0$ for all $B \in \mathcal{M} \cap \mathcal{P}(A)$.

(Key) Example: Let f be an integrable function on a measure space $(X, \mathcal{M}, \mu)$.
Define $\nu(A) := \int_A f \, d\mu$, for $A \in \mathcal{M}$. Then, ν is a signed measure on $\mathcal{M}$.
If $P := \{x \in X : f(x) \geq 0\}$ and $N := \{x \in X : f(x) < 0\}$, then P is a positive set and N is a negative set for ν.

Prop. Let μ be a signed measure on $\mathcal{M}$, and let $E \in \mathcal{M}$ be s.t. $\mu(E) < 0$. Then, there exists a negative set $F \subset E$ with $\mu(F) < 0$.

Pf. If E itself is a negative set, there is nothing to show. So, suppose $\exists B \in \mathcal{M}$ s.t. $B \subset E$ and $\mu(B) > 0$. Let $n_1 \in \mathbb{Z}_+$ be minimum s.t. $\exists B \in \mathcal{M}, B \subset E$ with $\mu(B) \geq \frac{1}{n_1}$.
Choose $E_1 \in \mathcal{M} \cap \mathcal{P}(E)$ s.t. $\mu(E_1) \geq \frac{1}{n_1}$. Inductively, we define a sequence $(E_i)_{i=1}^{\infty} \subset \mathcal{M} \cap \mathcal{P}(E)$ as follows: Given pairwise disjoint $E_1, E_2, \dots, E_k$ with $\mu(E_i) > 0$,
set $F_{k+1} := E \setminus \left(\bigcup_{i=1}^{k} E_i\right)$. If F_{k+1} is a negative set, then

$$\mu(F_{k+1}) = \mu\left(E \setminus \left(\bigcup_{i=1}^{k} E_i\right)\right) \overset{\text{b/c } \mu(E) < +\infty}{=} \mu(E) - \sum_{i=1}^{k} \mu(E_i) < \mu(E) < 0.$$

Letting $F := F_{k+1}$, we're done.
Otherwise (if F_{k+1} is not a negative set), let $n_{k+1} \in \mathbb{Z}_+$ be the minimum s.t. $\exists E_{k+1} \in \mathcal{M}$, $E_{k+1} \subset F_{k+1}$ with $\mu(E_{k+1}) \geq \frac{1}{n_{k+1}}$. This defines E_{k+1} so that $\{E_1, \dots, E_{k+1}\}$ are pairwise

disjoint, measurable, and $\mu(E_i) > 0$, $i = 1, \dots, k+1$.

Suppose the above process doesn't terminate. Then, set $F := \bigcap_{k=2}^{\infty} F_k \left(= E \setminus \left(\bigcup_{i=1}^{\infty} E_i\right)\right)$.

We claim that F is the required negative set.

Since $-\infty < \mu(E) < 0$ and $\mu(E_i) > 0, \forall i$, we have $\mu\left(E \setminus \left(\bigcup_{i=1}^{\infty} E_i\right)\right) = \lim_{n\to\infty} \mu\left(E \setminus \bigcup_{i=1}^{n} E_i\right)$,

hence $\mu(F) = \lim_{n\to\infty}\left[\mu(E) - \mu\left(\bigcup_{i=1}^{n} E_i\right)\right] = \mu(E) - \lim_{n\to\infty}\left(\sum_{i=1}^{n}\mu(E_i)\right) = \mu(E) - \sum_{i=1}^{\infty}\mu(E_i)$.

It follows that $\mu(F) \le \mu(E) < 0$, and the series $\sum_{i=1}^{\infty}\mu(E_i)$ converges.

Since $\mu(E_i) \ge \frac{1}{n_i}$, we have $n_i \xrightarrow[i\to\infty]{} \infty$. Suppose F is not a negative set, and let $G \subset F$, $G \in \mathcal{M}$ be s.t. $\mu(G) > 0$. Then $\mu(G) \ge \frac{1}{N}$ for some $N \in \mathbb{Z}_+$, which contradicts the construction: Indeed, let $i_0 \in \mathbb{Z}_+$ be the min. s.t. $n_{i_0} > N$. Then, at the i_0'th step of the above construction we should have chosen N instead of n_{i_0}, since there exists a measurable set in $E \setminus \bigcup_{i=1}^{i_0-1} E_i$ with measure $\ge \frac{1}{N}$ (namely G). This contradiction completes the proof. $\square$

<u>Thm. (Hahn Decomposition Thm.)</u> Let $\mu : \mathcal{M} \to (-\infty, \infty]$ be a signed measure (on a σ-alg. $\mathcal{M}$ on X).

(1) There exists $E \in \mathcal{M}$ s.t. E is a negative set and $F := E^c$ is a positive set.

(2) If $\{E', F'\}$ is another such pair, then $E \triangle E' = F \triangle F'$ is a null set of μ.

(3) If μ is not a positive measure then $\mu(E) < 0$. If $-\mu$ is not a positive measure then $\mu(F) > 0$.

Pf. (1) If $\mathcal{M}$ contains no non-empty negative set for μ, then by the last proposition, $\mu(A) \ge 0$, $\forall A \in \mathcal{M}$ and there is nothing to prove. Otherwise, let $L := \inf\{\mu(A) : A \text{ a negative set for } \mu\}$. Choose $(A_n)_{n=1}^{\infty} \subset \mathcal{M}$ a sequence of negative sets s.t. $\mu(A_n) \xrightarrow[n\to\infty]{} L$, and set $E := \bigcup_{n=1}^{\infty} A_n$.

We claim that E is the required negative set.

Indeed, let $B_1 := A_1$ and, $\forall n \in \mathbb{Z}_+$, $B_{n+1} := A_{n+1} \setminus \left(\bigcup_{i=1}^{n} B_i\right)$. Then, $\{B_n\} \subset \mathcal{M}$ and each B_n is a negative set as a subset of a negative A_n. We have $B_i \cap B_j = \phi$, $\forall i \ne j$, and $\bigcup_n B_n = \bigcup_n A_n = E$.

Hence, for any $C \subset E$, $\mu(C) = \mu\left(C \cap \left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \lim_{n\to\infty} \mu\left(C \cap \left(\bigcup_{i=1}^{n} B_i\right)\right) = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} \mu(C \cap B_i)}_{\le 0, \forall n} \le 0$.

This shows that E is negative.

Moreover, $\forall n$, $\mu(E) = \mu(A_n) + \overbrace{\mu(E \setminus A_n)}^{\le 0, \text{ b/c } E \text{ negative}} \le \mu(A_n)$, and hence $\mu(E) \le L$. By def. of L, it follows that $\mu(E) = L$.

Next, we claim that $F := E^c$ is positive. Suppose otherwise, and let $G \subset F$, $G \in \mathcal{M}$ be s.t.

$\mu(G)<0$. By the proposition, G contains a negative set H with $\mu(H)<0$. Then, $H\subset G\subset E^c$, so $\mu(H\cup E)=\mu(H)+\mu(E)$. The latter implies that $\mu(H\cup E)<L$, which contradicts the def'n of L. ✓

(2) Let E', $F'=(E')^c$ be s.t. E' negative, F' positive. We have $E\Delta E'=(E\setminus E')\cup(E'\setminus E)=$

$$=(E\cap(E')^c)\cup(E'\cap E^c)=((F^c)^c\cap F')\cup(((F')^c)^c\cap F)=(F'\setminus F)\cup(F\setminus F')=F\Delta F'.$$

For any measurable $A_1\subset E\setminus E'$, we have $A_1\subset E\setminus E'=F'\setminus F\subset F'$, so $\mu(A_1)=0$. Similarly, for any m'ble $A_2\subset E'\setminus E$, $A_2\subset E'\setminus E=F\setminus F'\subset F$, so $\mu(A_2)=0$. Since any $A\subset E\Delta E'$ is the union of A_1, A_2 as above, we have $\mu(A)=0$. ✓

(3) Suppose μ is not positive but $\mu(E)=0$. Then, for any $A\in\mathcal{M}$, we have

$$\mu(A)=\mu((A\cap E)\sqcup(A\cap F))=\mu(A\cap E)+\mu(A\cap F)=(\mu(E)\underbrace{-\mu(E\setminus A)}_{\leq 0,\text{ b/c } E \text{ is negative}})+\mu(A\cap F)\geq \overset{0}{\mu(E)}+\mu(A\cap F)\geq 0.$$

This means that μ is positive; a contradiction.

The proof for $-\mu$ and F is analogous. □

Def. Let $(X,\mathcal{M})$ be a measurable space. Two (positive) measures μ,ν are <u>mutually singular</u> when $\exists E\in\mathcal{M}$ s.t. $\mu(E)=\nu(E^c)=0$. We write $\mu\perp\nu$.

<u>Examples</u>: 1) $(X,\mathcal{M}):=([0,1],\mathcal{L}^1)$, $\mu(A):=m(A\cap[0,\frac12])$, $\nu(A):=m(A\cap[\frac12,1])$.

2) $X=\mathbb{R}$, $\mathcal{M}=\mathcal{L}^1$, $f:X\to[0,1]$ defined as $f|_{(-\infty,0]}\equiv 0$, $f|_{[1,\infty)}\equiv 1$, $f|_{[0,1]}=$ Cantor-Lebesgue f'n. Set $\mu:=m$, $\nu:=$ Lebesgue-Stieltjes measure rel. f. Then $\mu\perp\nu$, b/c $\mu(\text{Cantor set})=0$ and $\nu(C^c)=0$, as f is locally constant on C^c (i.e., constant on every connected comp. of C^c).

<u>Thm. (Jordan Decomposition Thm.)</u> If μ is a signed measure on a measurable space $(X,\mathcal{M})$, then there exist positive measures μ^+ and μ^- on $\mathcal{M}$, s.t. $\mu=\mu^+-\mu^-$ and $\mu^+\perp\mu^-$. This decomposition is unique.

Pf. Let E and $F:=E^c$ be a negative and a positive sets for μ, from the Hahn's thm. Set $\mu^+(A):=\mu(A\cap F)$ and $\mu^-(A):=-\mu(A\cap E)$, for any $A\in\mathcal{M}$. Then, μ^+ and μ^- are positive measures, $\mu=\mu^+-\mu^-$ (by additivity of μ), and $\mu^+(E)=\mu^-(F)=0$ with $E\sqcup F=X$. ✓

Suppose then that $\mu=\nu^+-\nu^-$ is another such decomposition, with $\nu^+\perp\nu^-$.

Let $E' \in \mathcal{M}$ be s.t. $\nu^+(E') = 0$ and $\nu^-(F') = 0$, where $F' := (E')^c$.

Now, for any $A \subset E'$, $\mu(A) = \nu^+(A) - \nu^-(A) \leq \nu^+(E') = 0$, so E' is a negative set for μ.

Similarly, $\forall A \subset F'$, $\mu(A) = \nu^+(A) - \nu^-(A) \overset{\text{as } A \subset F'}{=} \nu^+(A) \geq 0$, so F' is a positive set for μ.

Thus, by the Hahn thm., $E \Delta E' = F \Delta F'$ is a μ-null set. It follows that, $\forall A \in \mathcal{M}$,

$$\nu^+(A) = \nu^+(A \cap F') + \nu^+(A \cap E') \overset{\nu^+(E')=0}{=} \nu^+(A \cap F') \overset{\nu^-(F')=0}{=} \nu^+(A \cap F') - \nu^-(A \cap F') = \mu(A \cap F') \overset{\mu(F \Delta F')=0}{=} \mu(A \cap F) \overset{\text{def}^{\text{n}} \text{ of } \mu^+}{=} \mu^+(A).$$

Similarly, $\nu^-(A) = \nu^-(A \cap E') + \nu^-(A \cap F') \overset{\nu^-(F')=0}{=} \nu^-(A \cap E') \overset{\nu^+(E')=0}{=} \nu^-(A \cap E') - \nu^+(A \cap E') = -\mu(A \cap E') \overset{\mu(E \Delta E')=0}{=} -\mu(A \cap E) = \mu^-(A)$; so $\nu^{+/-} = \mu^{+/-}$. $\square$

Def. The measure $|\mu| := \mu^+ + \mu^-$ is called the <u>total variation measure</u> of μ, and $|\mu|(X)$ is the <u>total variation</u> of μ. The μ^+ and μ^- are called the <u>positive-</u> and <u>negative variations</u> of μ.

XI. COMPLEX MEASURES

Def. Let $\mathcal{M}$ be a σ-algebra on a set X. A **complex measure** on $\mathcal{M}$ is a function $\mu : \mathcal{M} \to \mathbb{C}$ satisfying $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$, where $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$, $E_i \cap E_j = \emptyset$, $\forall i \neq j$.

Remarks: 1) Since $\mu(E) \in \mathbb{C}$, $\forall E \in \mathcal{M}$, it follows that $\mu(\emptyset) = 0$. (Indeed, since $\emptyset = \emptyset \sqcup \emptyset$, $\mu(\emptyset) = 2\mu(\emptyset)$.)

2) For the same reason, the series $\sum \mu(E_n)$ is always convergent.
Also after any rearrangement, hence it is absolutely convergent.

(Key) Example: $(X, \mathcal{M}, \mu)$ σ-finite measure space, $h : X \to \mathbb{C}$ integrable function. Then,

$$\nu(A) := \int_A h \, d\mu, \quad A \in \mathcal{M} \qquad \text{is a complex measure.}$$

Def. Let $\mu : \mathcal{M} \to \mathbb{C}$ be a complex measure on a σ-algebra $\mathcal{M}$. The function $|\mu| : \mathcal{M} \to \mathbb{R}$ defined as $|\mu|(E) := \sup\left\{ \sum_{n=1}^{\infty} |\mu(E_n)| : \{E_n\}_{n=1}^{\infty} \subset \mathcal{M},\ E = \bigcup_{n=1}^{\infty} E_n,\ E_i \cap E_j = \emptyset,\ \forall i \neq j \right\}$, for $E \in \mathcal{M}$, is called the **total variation measure of μ**.

Remark: For any $A \in \mathcal{M}$, we have $|\mu|(A) \geq |\mu(A)|$.

Indeed, given $A \in \mathcal{M}$, the family $\{A, \emptyset, \emptyset, \dots\}$ forms a partition of A, so
$|\mu|(A) \geq |\mu(A)| + \sum_{n=2}^{\infty} |\mu(\emptyset)| = |\mu(A)|$.

Thm. The total variation $|\mu|$ of a complex measure μ on $\mathcal{M}$ is a positive measure on $\mathcal{M}$.

Pf. Let $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$ be arbitrary collection of pairwise disjoint sets. Set $E := \coprod_{n=1}^{\infty} E_n$.

We want to show that $|\mu|(E) = \sum_{n=1}^{\infty} |\mu|(E_n)$.

For every $n \in \mathbb{Z}_+$, let $t_n \in \mathbb{R}$ be s.t. $|\mu|(E_n) > t_n$. Then, $\forall n$, there exist pairwise disjoint $\{A_{nk}\}_{k=1}^{\infty} \subset \mathcal{M}$ s.t. $E_n = \bigcup_{k=1}^{\infty} A_{nk}$ and $\sum_{k=1}^{\infty} |\mu(A_{nk})| \geq t_n$. Since $\{A_{nk}\}_{n,k=1}^{\infty}$ is a partition of E, we have $|\mu|(E) \geq \sum_{n,k=1}^{\infty} |\mu(A_{nk})| = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} |\mu(A_{nk})| \right) \geq \sum_{n=1}^{\infty} t_n$. Taking supremum over all $(t_n)_{n=1}^{\infty}$ as above, we get $|\mu|(E) \geq \sum_{n=1}^{\infty} |\mu|(E_n)$.

For the opposite inequality, let $\{A_k\}_{k=1}^{\infty} \subset \mathcal{M}$ be an arbitrary partition of E. Then, $\forall k$, $\{A_k \cap E_n\}_{n=1}^{\infty}$ is a partition of A_k, and $\forall n$, $\{A_k \cap E_n\}_{k=1}^{\infty}$ is a partition of E_n,

so
$$\sum_{k=1}^{\infty} |\mu(A_k)| = \sum_{k=1}^{\infty} \left| \sum_{n=1}^{\infty} \mu(A_k \cap E_n) \right| \leq \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\mu(A_k \cap E_n)| \overset{\text{by absolute convergence, since } \{A_k \cap E_n\}_{k,n} \text{ is a partition of } E}{=} \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} |\mu(A_k \cap E_n)| \right) \leq \sum_{n=1}^{\infty} |\mu|(E_n).$$

Taking supremum over all such partitions $\{A_k\}$, we get $|\mu|(E) \leq \sum_{n=1}^{\infty} |\mu|(E_n)$.

This proves countable additivity of $|\mu|$. That $|\mu|(\emptyset) = 0$ follows from def'n and $\mu(\emptyset) = 0$. $\square$

Thm. If μ is a complex measure on $(X, \mathcal{M})$, then the measure $|\mu|$ is finite (i.e., $|\mu|(X) < \infty$)

We'll need the following:

Lemma. If $\{z_1, \dots, z_N\} \subset \mathbb{C}$, then $\exists S \subset \{1, \dots, N\}$ s.t. $\left| \sum_{n \in S} z_n \right| \geq \frac{1}{\pi} \sum_{n=1}^{N} |z_n|$.

Pf. Write $z_n = |z_n| \cdot e^{i\alpha_n}$, with $\alpha_n \in (-\pi, \pi]$. For $\theta \in [-\pi, \pi]$, let $S(\theta) := \{n \leq N : \cos(\alpha_n - \theta) > 0\}$ (i.e., $|\alpha_n - \theta| \in [0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi]$).

Then,
$$\left| \sum_{n \in S(\theta)} z_n \right| = |e^{-i\theta}| \cdot \left| \sum_{S(\theta)} z_n \right| = \left| \sum_{S(\theta)} e^{-i\theta} z_n \right| \geq \mathrm{Re}\left(\sum_{S(\theta)} |z_n| \cdot e^{i(\alpha_n - \theta)} \right) = \sum_{S(\theta)} |z_n| \cdot \cos(\alpha_n - \theta) = \sum_{n=1}^{N} |z_n| \cdot \cos^+(\alpha_n - \theta).$$

Choose $\theta_0 \in [-\pi, \pi]$ so as to maximize the last sum, and set $S := S(\theta_0)$.

Then,
$$\left| \sum_S z_n \right| \geq \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \left(\sum_{n=1}^{N} |z_n| \cdot \cos^+(\alpha_n - \theta) \right) d\theta = \sum_{n=1}^{N} \Big(|z_n| \cdot \frac{1}{2\pi} \underbrace{\int_{-\pi}^{\pi} \cos^+(\alpha_n - \theta)\, d\theta}_{\forall n:\ = \text{area under } \cos^+|_{-\pi}^{\pi} = 2 \cdot \sin(\frac{\pi}{2}) = 2} \Big) = \sum_{n=1}^{N} \left(|z_n| \cdot \frac{1}{\pi} \right) = \frac{1}{\pi} \cdot \sum_{n=1}^{N} |z_n|. \ \square$$

Pf. of Thm.: Suppose first that for some $E \in \mathcal{M}$ we have $|\mu|(E) = \infty$.

Set $t := \pi(1 + |\mu(E)|)$. Since $|\mu|(E) = \infty$, there is a partition $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$ of E s.t. $\sum_{n=1}^{N} |\mu(E_n)| > t$, for some $N \in \mathbb{Z}_+$. Apply the above lemma, with $z_n := \mu(E_n)$, to conclude that there is a set $A \subset E$, $A \in \mathcal{M}$ (namely, the union of some of the $E_1, \dots, E_N$) s.t. $|\mu(A)| > \frac{t}{\pi} \geq 1$. Setting $B := E \setminus A$, we get

$$|\mu(B)| = |\mu(E \setminus A)| = |\mu(E) - \mu(A)| \geq |\mu(A)| - |\mu(E)| > \frac{t}{\pi} - |\mu(E)| \overset{\text{by def'n of } t}{=} 1$$

We thus have $E = A \sqcup B$ with $|\mu(A)| > 1$ and $|\mu(B)| > 1$. By additivity of $|\mu|$, at least one of the A, B must have infinite $|\mu|$-measure.

Now, if $|\mu|(X) = \infty$, we construct an infinite sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{M}$ of pairwise disjoint sets with $|\mu(A_n)| > 1$, $\forall n$, as follows: Split X into A_1 and B_1 with $|\mu(A_1)| > 1$ and $|\mu|(B_1) = \infty$. Given $A_1, \dots, A_k$, split B_k into A_{k+1} and B_{k+1} with $|\mu(A_{k+1})| > 1$ and $|\mu|(B_{k+1}) = \infty$. Then, since the A_n are pairwise disjoint, we have

$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$, and $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$. But then the series $\sum_n \mu(A_n)$ must converge which is impossible, since $|\mu(A_n)| \underset{n\to\infty}{\not\to} 0$. □

<u>Remark</u>: If μ, ν are complex measures on $\mathcal{M}$, then so are $\mu+\nu$ and $c\mu$, for any $c \in \mathbb{C}$, where $(\mu+\nu)(E) := \mu(E) + \nu(E)$, $(c\mu)(E) = c \cdot \mu(E)$. Hence {complex measures on $\mathcal{M}$} is a $\mathbb{C}$-vector space. The function $\|\mu\| := |\mu|(X)$ is a norm on this space. <u>Exercise</u>!

<u>Absolute Continuity</u>

<u>Def.</u> Let μ be a positive measure on a σ-algebra $\mathcal{M}$, and let λ be an arbitrary measure on $\mathcal{M}$. We say that λ is <u>absolutely continuous w/r to μ</u>, and write $\lambda \ll \mu$, when $\lambda(E) = 0$ for all $E \in \mathcal{M}$ with $\mu(E) = 0$.

We say that λ is <u>concentrated</u> on a measurable set <u>A</u>, when $\lambda(E) = 0$, $\forall E \in \mathcal{M}$ s.t. $E \cap A = \emptyset$

<u>Prop.</u> Suppose $\lambda, \lambda_1, \lambda_2$ are measures on $\mathcal{M}$, and μ is a positive measure on $\mathcal{M}$. Then:

(1) If λ is concentrated on $A \in \mathcal{M}$, then so is $|\lambda|$.
(2) If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$.
(3) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $(\lambda_1 + \lambda_2) \perp \mu$.
(4) If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $(\lambda_1 + \lambda_2) \ll \mu$
(5) If $\lambda \ll \mu$, then $|\lambda| \ll \mu$.
(6) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.
(7) If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda \equiv 0$.

Pf. (1) For any $E \subset A^c$, $E \in \mathcal{M}$ and any partition $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$ of E, we have $E_n \cap A = \emptyset, \forall n$, hence $\sum_{n=1}^{\infty} |\lambda(E_n)| = \sum_n 0 = 0$. Thus, $|\lambda|(E) = 0$. ✓

(2) Follows from (1). ✓

(3) There are $A_1, B_1, A_2, B_2 \in \mathcal{M}$ s.t. $A_1 \cap B_1 = A_2 \cap B_2 = \emptyset$, λ_1 is concentrated on A_1, μ is concentrated on B_1, λ_2 is concentrated on A_2, and μ is concentrated on B_2. Then, $\lambda_1 + \lambda_2$ is concentrated on $A_1 \cup A_2$, and μ is concentrated on $B_1 \cap B_2$ (indeed, if $E \subset (B_1 \cap B_2)^c$ then $E = E_1 \cup E_2$, where $E_i = E \cap B_i^c$; then $\mu(E_1) = \mu(E_2) = 0$, so $\mu(E) = 0$) But $(A_1 \cup A_2) \cap (B_1 \cap B_2) = \emptyset$. ✓

(4) Clear. ✓

(5) Suppose $E \in \mathcal{M}$ is s.t. $\mu(E) = 0$. Then, $\mu(A) = 0$, $\forall A \in \mathcal{M} \cap \mathcal{P}(E)$. Hence, for any partition $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$ of E, we have $\sum_{n=1}^{\infty} |\lambda(E_n)| \overset{\lambda \ll \mu}{=} \sum_n 0$, so $|\lambda|(E) = 0$. ✓

(6) Let $A \in \mathcal{M}$ be s.t. $\mu(A) = 0$ and λ_2 is concentrated on A. Since $\lambda_1 \ll \mu$, it follows that $\forall E \in \mathcal{M} \cap \mathcal{P}(A)$, $\lambda(E) = 0$, and so λ is concentrated on A^c. ✓

(7) By (6), we have $\lambda \perp \lambda$, which means that $\exists A \in \mathcal{M}$ s.t. $\lambda(E) = 0$, $\forall E \in \mathcal{M} \cap \mathcal{P}(A)$ and $\lambda(F) = 0$, $\forall F \in \mathcal{M} \cap \mathcal{P}(A^c)$. Thus, $\lambda(E) = 0$, $\forall E \in \mathcal{M}$. $\square$

<u>Thm. (Radon-Nikodym Thm.)</u> Let μ be a positive σ-finite measure on a σ-algebra $\mathcal{M}$ on X, and let λ be a complex measure on $\mathcal{M}$. Then:

(1) There is a unique pair of complex measures λ_a and λ_s on $\mathcal{M}$ s.t.

$$\lambda = \lambda_a + \lambda_s, \quad \lambda_a \ll \mu, \quad \lambda_s \perp \mu.$$

If λ is positive finite, then so are λ_a and λ_s.

(2) There is a unique (a.e.) integrable function $h: X \to \mathbb{C}$ s.t.

$$\lambda_a(E) = \int_E h \, d\mu, \quad \text{for all } E \in \mathcal{M}.$$

<u>Def.</u> The pair (λ_a, λ_s) is called the <u>Lebesgue decomposition</u> of λ rel. to μ.

Proof of the theorem in case λ = positive finite - in-class presentation.

Pf. of Radon-Nikodym:

Suppose first that λ and μ are finite positive measures.

Let $\mathcal{F} := \{ g : X \to [0,\infty) \mid g \text{ is } \mathcal{M}\text{-m'ble} \wedge \int_A g\,d\mu \le \lambda(A),\ \forall A \in \mathcal{M} \}$.

Note that $\mathcal{F} \neq \emptyset$, since $g \equiv 0 \in \mathcal{F}$.

Let $L := \sup \{ \int_X g\,d\mu \mid g \in \mathcal{F} \}$, and let $(g_n) \subset \mathcal{F}$ be s.t. $\lim_{n\to\infty} \int_X g_n\,d\mu = L$.

For $n \in \mathbb{Z}_+$, define $h_n := \max\{g_1, \dots, g_n\}$. Then, $\forall n$, $h_n : X \to [0,\infty)$ is $\mathcal{M}$-m'ble and $h_n \in \mathcal{F}$. (Indeed, by induction it suffices to prove that $g_1, g_2 \in \mathcal{F} \Rightarrow h = \max\{g_1, g_2\} \in \mathcal{F}$, but $\int_A h\,d\mu = \int_{A\cap B} h\,d\mu + \int_{A\cap B^c} h\,d\mu = \int_{A\cap B} g_1\,d\mu + \int_{A\cap B^c} g_2\,d\mu \le \lambda(A\cap B) + \lambda(A \cap B^c) = \lambda(A)$, $\forall A \in \mathcal{M}$, where $B = \{x \in X \mid g_1(x) \ge g_2(x)\}$.)

Let $h := \lim_{n\to\infty} h_n$. Then, $h_n \nearrow h$, so h is $\mathcal{M}$-m'ble, and by Monotone Conv. Thm,
$\int_X h\,d\mu = \lim_{n\to\infty} \int_X h_n\,d\mu$.

For any $A \in \mathcal{M}$, we have $\int_A h_n\,d\mu \le \lambda(A)$, hence also $\int_A h\,d\mu \le \lambda(A)$. Thus, $h \in \mathcal{F}$.

On the other hand, $\forall n$, $h \ge g_n$, so $\int_X h\,d\mu \ge \int_X g_n\,d\mu$, hence $\int_X h\,d\mu \ge \lim_{n\to\infty} \int_X g_n\,d\mu = L$.

Thus, $\int_X h\,d\mu = L$.

Define $\lambda_a(A) := \int_A h\,d\mu$, $\forall A \in \mathcal{M}$. Then, define $\lambda_s := \lambda - \lambda_a$. ($\lambda_s$ is positive, since $\forall A \in \mathcal{M}$, $\int_A h\,d\mu \le \lambda(A)$)

Clearly, $\lambda_a \ll \mu$. We claim that also $\lambda_s \perp \mu$.

<u>Lemma</u>. Let ν, μ be finite positive measures on $\mathcal{M}$. Then, either $\nu \perp \mu$ or else $\exists \varepsilon > 0$ $\exists G \in \mathcal{M}$ s.t. $\mu(G) > 0$ and G is a positive set for $\nu - \varepsilon\mu$.

Pf.

For $n \in \mathbb{Z}_+$, let $\{E_n, F_n\}$ be a Hahn decomposition of X for $\nu - \frac{1}{n}\mu$.

Set $E := \bigcap_{n=1}^{\infty} E_n$, $F := \bigcup_{n=1}^{\infty} F_n$. Then, $E^c = F$, and since $\forall n$, $E \subset E_n$, we have

$\nu(E) \le \nu(E_n) \le \frac{1}{n}\mu(E_n) \le \frac{1}{n}\mu(X)$. Since $\mu(X) < \infty$, it follows that $\nu(E) = 0$.

If now $\mu(E^c) = 0$, then $\nu \perp \mu$. If, in turn, $\mu(F) = \mu(E^c) > 0$, then $\exists n_0 \ge 1$ s.t. $\mu(F_{n_0}) > 0$.

In this case, we let $G := F_{n_0}$ and $\varepsilon = 1/n_0$. ∎

For a proof by contradiction, suppose $\lambda_s \not\perp \mu$. Then, by Lemma, $\exists \varepsilon > 0$ $\exists G \in \mathcal{M}$ s.t. $\mu(G) > 0 \wedge G$ is positive for $\lambda_s - \varepsilon\mu$. Hence, $\forall A \in \mathcal{M}$,

$\lambda(A) - \int_A h\,d\mu = \lambda(A) - \lambda_a(A) = \lambda_s(A) \ge \lambda_s(A \cap G) \ge \varepsilon\mu(A \cap G) = \int_A \varepsilon \cdot \chi_G\,d\mu$, and so

$\lambda(A) \ge \int_A (h + \varepsilon\chi_G)\,d\mu$. It follows that $h + \varepsilon\chi_G \in \mathcal{F}$, and so $\int_X (h + \varepsilon\chi_G)\,d\mu \le L$.

But $\int_X (h + \varepsilon\chi_G)\,d\mu = \int_X h\,d\mu + \varepsilon\mu(G) = L + \varepsilon\mu(G) > L$. ↯

Let now μ be σ-finite, and let $\{X_n\}_{n=1}^{\infty} \subset \mathcal{M}$ be s.t. $X_n \uparrow X$ and $\mu(X_n) < \infty$, $\forall n$.

Define, $\forall n \in \mathbb{Z}_{+}$, $\mu_n(A) := \mu(A \cap X_n)$, $\forall A \in \mathcal{M}$, and $\lambda_n(A) := \lambda(A \cap X_n)$

By the above, $\forall n \in \mathbb{Z}_{+}$, $\exists h_n : X_n \to [0,\infty)$ $\exists (\lambda_n)_a, (\lambda_n)_s$ positive s.t. $\lambda_n = (\lambda_n)_a + (\lambda_n)_s$, $(\lambda_n)_a = \int_{X_n} h_n \, d\mu_n$, and $(\lambda_n)_s \perp \mu_n$.

By uniqueness of $h_n, (\lambda_n)_a, (\lambda_n)_s$, it follows that

$$h_{n+1}\big|_{X_n} = h_n \ \text{(a.e.)}, \text{ and } (\lambda_{n+1})_a\big|_{\mathcal{B}(X_n)} = (\lambda_n)_a, \ (\lambda_{n+1})_s\big|_{\mathcal{B}(X_n)} = (\lambda_n)_s .$$

Hence, can define $h : X \to [0,\infty)$ as $h(x) := h_n(x)$ if $x \in X_n$. Set $\lambda_a(A) := \int_A h \, d\mu$.

Then, $\forall n \geq 1$ $\forall A \in \mathcal{M}$, $\lambda_a(A) = \lim_{n\to\infty} \lambda_a(A \cap X_n) = \lim_{n\to\infty} \int_{A\cap X_n} h_n \, d\mu_n = \lim_{n\to\infty} \int_X h_n \cdot \chi_{A\cap X_n} \, d\mu \overset{MCT(!)}{=}$

$$= \int_X \lim_{n\to\infty} (h_n \chi_{A\cap X_n}) \, d\mu = \int_X h \cdot \chi_A \, d\mu = \int_A h \, d\mu .$$

Finally, if, $\forall n \geq 1$, $E_n \in \mathcal{M} \cap \mathcal{B}(X_n)$ is s.t. $(\lambda_n)_s(E_n) = 0 = \mu_n(E_n^c)$,

then setting $E := \bigcap_{n=1}^{\infty} E_n$, we get $\mu(E^c) = \mu\left(\bigcup_{n=1}^{\infty} E_n^c\right) \leq \sum_{n=1}^{\infty} \mu(E_n^c) = \sum_{n=1}^{\infty} \mu_n(E_n^c) = \sum 0 = 0$,

and $\lambda_s(E) \leq \lambda_s(E_1) = (\lambda_1)_s(E_1) = 0$, which proves $\lambda_s \perp \mu$ (provided λ_s is positive).

The latter follows from:

$$\lambda(A) \geq \lambda(A \cap X_n) = \lambda_n(A \cap X_n) \geq \int_{A\cap X_n} h_n \, d\mu_n = \int_X h_n \cdot \chi_{A\cap X_n} \, d\mu, \quad \forall n \geq 1 \quad (\forall A \in \mathcal{M})$$

hence $\lambda(A) \geq \lim_{n\to\infty} \int_X h_n \chi_{A\cap X_n} \, d\mu \overset{MCT}{=} \int_A h \, d\mu$.

Finally, if $\lambda : \mathcal{M} \to \mathbb{C}$ is a cplx measure, write $\lambda = \lambda_r + i \cdot \lambda_i$, where λ_r (resp. λ_i) is the real (resp. imaginary) part of λ, and apply the above to $\lambda_r^+, \lambda_r^-, \lambda_i^+, \lambda_i^-$.