# XII. $L^p$ SPACES

Recall that a function $\varphi : (a,b) \to \mathbb{R}$ is called <u>convex</u> when

$$\varphi((1-\lambda)x + \lambda y) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y), \quad \text{for all } x,y \in (a,b),\ \lambda \in [0,1].$$

Equivalently, $\varphi$ is convex when $\dfrac{\varphi(t)-\varphi(s)}{t-s} \leq \dfrac{\varphi(u)-\varphi(t)}{u-t}$, for all $a<s<t<u<b$.

<u>Remarks</u>:

1) If $\varphi$ is differentiable on $(a,b)$, then $\varphi$ convex on $(a,b)$ iff $\varphi'$ is increasing on $(a,b)$. /by MVT/

2) $\varphi$ convex on $(a,b) \Rightarrow \varphi$ cont's on $(a,b)$. /<u>Exercise</u>/

<u>Thm. (Jensen's Inequality)</u> Let $(X, \mathcal{M}, \mu)$ be a measure space, with $\mu(X)=1$. Suppose $f : X \to (a,b)$ is integrable and $\varphi : (a,b) \to \mathbb{R}$ is convex. Then, $\varphi \circ f$ is measurable and

$$\varphi\left(\int_X f \, d\mu\right) \leq \int_X (\varphi \circ f)\, d\mu.$$

Pf. Since $f$ measurable and $\varphi$ cont's, it follows that $\varphi \circ f$ measurable.

Set $t := \int_X f\,d\mu$. Since $\mu(X)=1$ and $a < f(x) < b$, $\forall x \in X$, we have $a<t<b$.

Let $\beta := \sup_{a<s<t} \dfrac{\varphi(t)-\varphi(s)}{t-s}$. Then, $\beta \leq \dfrac{\varphi(u)-\varphi(t)}{u-t}$ for all $t<u<b$.

Thus, for every $y \in (a,t)$, $\varphi(t)-\varphi(y) \leq \beta(t-y)$, and

for every $y \in (t,b)$, $\varphi(y) - \varphi(t) \geq \beta(y-t)$, so $\quad \varphi(y) \geq \varphi(t) + \beta(y-t)$, $\forall y \in (a,b)$.

In particular, letting $y = f(x)$, $x \in X$, we get $\varphi(f(x)) \geq \varphi(t) + \beta(f(x)-t)$.

Integrating over $X$ gives $\int_X (\varphi\circ f)\,d\mu \geq \mu(X)\cdot\varphi(t) + \beta\cdot\int_X f\,d\mu - \mu(X)\cdot\beta t = \varphi\left(\int_X f\,d\mu\right) + \cancel{\beta\int_X f\,d\mu} - \cancel{\beta\int_X f\,d\mu}$. $\square$

<u>Def.</u> If $p, q \in (0,\infty)$ are such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$ (or, $p+q = pq$), then $p$ and $q$ are called a pair of <u>conjugate exponents</u>.

<u>Remark</u> Note that if $p,q$ are conjugate exponents, then $p,q>1$.

Letting $p \to 1^+$, we get $q \to \infty$, which is why $\{1,\infty\}$ are also called conjugate exp's.

**Thm.** Let $p$ and $q$ be conjugate exponents, $1<p,q<\infty$. Let $(X,\mathcal{M},\mu)$ be a measure space, and let $f,g: X\to[0,\infty]$ be measurable. Then:

(1) $\displaystyle\int_X fg\,d\mu \le \left(\int_X f^p\,d\mu\right)^{1/p}\cdot\left(\int_X g^q\,d\mu\right)^{1/q}$ /"Hölder's inequality" ["Schwarz" if $p=q=2$]/

(2) $\displaystyle\left(\int_X (f+g)^p\,d\mu\right)^{1/p} \le \left(\int_X f^p\,d\mu\right)^{1/p}+\left(\int_X g^p\,d\mu\right)^{1/p}$ /"Minkowski inequality"/

Pf. (1) Set $A:=\left(\int_X f^p\,d\mu\right)^{1/p}$ and $B:=\left(\int_X g^q\,d\mu\right)^{1/q}$.

If $A=0$, then $f=0$ a.e., hence so is $fg$ and the inequality holds.

If $A>0$ and $B=\infty$, then RHS $=\infty$, so again there is nothing to show.

We may thus assume that $0<A,B<\infty$. Put $F(x):=\frac{f(x)}{A}$, $G(x):=\frac{g(x)}{B}$, for $x\in X$.

Then, $\int_X F^p\,d\mu=\int_X G^q\,d\mu=1$. It follows that $\underbrace{\{x\in X: F(x)=\infty\}}_{=:S_f}$ and $\underbrace{\{x\in X: G(x)=\infty\}}_{=:S_g}$ have measure $0$.

For any $x\in(S_f)^c\cap(S_g)^c \cap (F^{-1}(0))^c\cap(G^{-1}(0))^c$, there are $s=s(x)$ and $t=t(x)$ s.t. $F(x)=e^{s/p}$ and $G(x)=e^{t/q}$.

Since $\frac{1}{p}+\frac{1}{q}=1$, the convexity of $e^x$ implies: $e^{s/p+t/q}\le\frac{1}{p}e^s+\frac{1}{q}e^t$.

It follows that $F(x)\cdot G(x)\le\frac{1}{p}(F(x))^p+\frac{1}{q}\cdot(G(x))^q$, for all $x\in(S_f\cup S_g)^c$.

Hence, $\displaystyle\int_X FG=\int_{(S_f\cup S_g)^c}FG\le\frac{1}{p}\int_{(S_f\cup S_g)^c}F^p+\frac{1}{q}\int_{(S_f\cup S_g)^c}G^q=\frac{1}{p}\int_X F^p+\frac{1}{q}\int_X G^q=\frac{1}{p}+\frac{1}{q}=1$, which proves (1).

For the proof of (2), write $(f+g)^p=f\cdot(f+g)^{p-1}+g\cdot(f+g)^{p-1}$.

By (1), we have $\int f\cdot(f+g)^{p-1}\le\left(\int f^p\right)^{1/p}\cdot\left(\int(f+g)^{(p-1)q}\right)^{1/q}$ and $\int g\cdot(f+g)^{p-1}\le\left(\int g^p\right)^{1/p}\cdot\left(\int(f+g)^{(p-1)q}\right)^{1/q}$.

Since $(p-1)q=pq-q=p$, we get

$$\int(f+g)^p\le\left(\int(f+g)^p\right)^{1/q}\cdot\left[\left(\int f^p\right)^{1/p}+\left(\int g^p\right)^{1/p}\right]. \quad (*)$$

W.l.o.g., we may assume that $\int f^p<\infty$ and $\int g^p<\infty$ (for else (2) holds trivially).

Then, $\int(f+g)^p<\infty$, since $\left(\frac{f+g}{2}\right)^p\le\frac{1}{2}(f^p+g^p)$ by convexity of $\{x\mapsto x^p\}$.

We may thus divide $(*)$ by $\left(\int(f+g)^p\right)^{1/q}$ and use $1-\frac{1}{q}=\frac{1}{p}$ to get

$$\left(\int(f+g)^p\right)^{1/p}\le\left(\int f^p\right)^{1/p}+\left(\int g^p\right)^{1/p}. \qquad \square$$

**Def.** Let $(X, \mathcal{M}, \mu)$ be a measure space, let $p \in [1, \infty)$, and let $f: X \to \mathbb{C}$ be a measurable function. Define $\|f\|_p := \left( \int_X |f|^p d\mu \right)^{1/p}$, called the <u>$L^p$-norm of $f$</u>, and $L^p(\mu) := \{ f : X \to \mathbb{C} \mid f \text{ measurable}, \|f\|_p < \infty \}$.

If $(X, \mathcal{M}, \mu) = (\mathbb{R}^n, \mathcal{L}_n, m_n)$, we write $L^p(\mathbb{R}^n)$ for $L^p(\mu)$.

If $\mu$ is the counting measure on $X$, we write $\ell^p(X)$ for $L^p(\mu)$.

**Def.** Let $(X, \mathcal{M}, \mu)$ be a measure space, and $g: X \to [0, \infty]$ a measurable function.
Let $S := \{ \alpha \in \mathbb{R} : \mu(\{x \in X : g(x) > \alpha\}) = 0 \}$. We define the <u>essential supremum</u> of $g$ as

$$\operatorname{ess\,sup} g := \begin{cases} \infty, & \text{if } S = \emptyset \\ \inf S, & \text{if } S \neq \emptyset. \end{cases}$$

For a measurable $f: X \to \mathbb{C}$, define $\|f\|_\infty := \operatorname{ess\,sup} |f|$.

The set $L^\infty(\mu) := \{ f : X \to \mathbb{C} \mid f \text{ measurable}, \|f\|_\infty < \infty \}$ is the set of <u>essentially bounded measurable functions</u>.

<u>Remarks.</u> 1) Suppose $\operatorname{ess\,sup} g =: \beta < \infty$.

Then, since $g^{-1}((\beta, \infty]) = \bigcup_{n=1}^{\infty} g^{-1}((\beta + \frac{1}{n}, \infty])$ and each $g^{-1}((\beta + \frac{1}{n}, \infty])$ has measure $0$ (by defn of $\beta$), it follows that $\mu(g^{-1}((\beta, \infty])) = 0$, and so $\beta \in S$.

2) We have: $\|f\|_\infty \leq \lambda \Leftrightarrow |f(x)| \leq \lambda$ for almost all $x$.

**Thm.** If $p$ and $q$ are conjugate exponents, $1 \leq p, q \leq \infty$, and if $f \in L^p(\mu)$, $g \in L^q(\mu)$, then $fg \in L^1(\mu)$ and $\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q$.

Pf. For $p > 1$, this is just the Hölder inequality (applied to $|f|$ and $|g|$).
If $p = \infty$, then $|f(x) \cdot g(x)| \leq \|f\|_\infty |g(x)|$ for almost all $x$ (by Remark (2)).
Thus, $\|fg\|_1 = \int |fg| \, d\mu \leq \int \|f\|_\infty |g| \, d\mu = \|f\|_\infty \int |g| \, d\mu = \|f\|_\infty \|g\|_1$, as required.
The case $p = 1$, $q = \infty$ is analogous. $\square$

**Thm.** Suppose $1 \leq p \leq \infty$, and $f, g \in L^p(\mu)$. Then, $f + g \in L^p(\mu)$ and $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

Pf. For $1 < p < \infty$, this follows from Minkowski's inequality, since $|f+g|^p \leq (|f| + |g|)^p$.
For $p = 1$, $\int |f+g| \, d\mu \leq \int |f| + |g| \, d\mu = \int |f| \, d\mu + \int |g| \, d\mu$.
For $p = \infty$, $|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty$ for almost all $x$, hence $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$ by Remark (2) again. $\square$

**Corollary.** $L^p(\mu)$ regarded as a set of equivalence classes modulo equality a.e. (rel. to $\mu$) is a normed complex vector space with the norm $\|\cdot\|_p$.

**Thm.** $L^p(\mu)$ is a complete metric space, for any $1 \le p \le \infty$ and positive measure $\mu$.

Pf. Assume first that $1 \le p < \infty$. Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $L^p(\mu)$.
We show first that $(f_n)$ contains a subsequence convergent pointwise a.e.
Indeed, for any $i \in \mathbb{Z}_+$, choose $n_i \, (> n_{i-1})$ s.t. $\|f_{n_{i+j}} - f_{n_i}\|_p < \frac{1}{2^i}$ for all $j \in \mathbb{Z}_+$.
Set $g_k := \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|$, and $g := \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|$. The Minkowski inequality and imply that $\|g_k\|_p < 1$, $\forall k$, and hence by Fatou's Lemma applied to the sequence $(g_k^{\,p})_{k=1}^{\infty}$ one gets $\|g\|_p \le 1$. In particular, $g(x) < \infty$ a.e., so the series $f_{n_1}(x) + \sum_{i=1}^{\infty} (f_{n_{i+1}}(x) - f_{n_i}(x))$ converges absolutely for almost every $x \in X$, and thus $f(x) := \lim_{i \to \infty} f_{n_i}(x)$ exists a.e.
Define $f(x) = 0$ on the remaining set of measure $0$.
We now want to show that $f \in L^p(\mu)$ and $f_n \xrightarrow[n\to\infty]{} f$ in $L^p(\mu)$.
Choose $\varepsilon > 0$. Let $N \in \mathbb{Z}_+$ be s.t. $\|f_m - f_n\|_p < \varepsilon$ for all $m, n \ge N$. Since $f = \lim_{i\to\infty} f_{n_i}$ a.e., it follows that $\int_X |f - f_N|^p \, d\mu \le \liminf_{i\to\infty} \int_X |f_{n_i} - f_N|^p \, d\mu \le \varepsilon^p$, by Fatou.
Thus, $f - f_N \in L^p(\mu)$, and hence $f = f_N + (f - f_N) \in L^p(\mu)$, by the previous theorem.
Also, the above inequalities show that $\|f - f_n\|_p \xrightarrow[n\to\infty]{} 0$. ✓

Suppose finally that $p = \infty$. Let $(f_n)$ be a Cauchy sequence in $L^\infty(\mu)$, and let $A_k = \{x \in X : |f_k(x)| > \|f_k\|_\infty\}$, $B_{m,n} = \{x \in X : |f_m(x) - f_n(x)| > \|f_m - f_n\|_\infty\}$, for $k, m, n \in \mathbb{Z}_+$.
Then, by def'n of the $L^\infty$-norm, all the $A_k$ and $B_{m,n}$ are of measure $0$, hence so is $E := \bigcup_k A_k \cup \bigcup_{m,n} B_{m,n}$. Now, on $X \setminus E$, the sequence $(f_n)$ converges uniformly to a bdd function $f$. Setting $f(x) := 0$ for $x \in E$ gives a function $f \in L^\infty(\mu)$ with $f_n \xrightarrow[n\to\infty]{} f$ in $L^\infty(\mu)$. □

**Lemma.** If $(X,\mathcal{M},\mu)$ is a $\sigma$-finite measure space, then there exists a function $w \in L^1(\mu)$ such that $w(x) \in (0,1)$, $\forall x \in X$.

Pf. If $\mu(X) < \infty$, there is nothing to show. Otherwise, let $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ be s.t. $\mu(E_n) < \infty$, $\forall n$ and $\bigcup_{n=1}^{\infty} E_n = X$. For $n \in \mathbb{Z}_+$, set $w_n := \frac{1}{2^n(1+\mu(E_n))} \cdot \chi_{E_n}$. Then, $\forall n$, $w_n\big|_{E_n} \in (0,1)$.
Set $w := \sum_{n=1}^{\infty} w_n$. Then, $w = \lim_{N\to\infty}\left(\sum_{n=1}^{N} w_n\right)$ is measurable; $\forall x \in X$, $w(x) \in (0,1)$ (since $\exists n : x \in E_n$, and $\sum_{n=1}^{\infty} \frac{\chi_{E_n}(x)}{2^n(1+\mu(E_n))} < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$). Moreover, by M.C.T., $\int_X w \, d\mu = \lim_{N\to\infty} \int_X \sum_{n=1}^{N} w_n \, d\mu \le \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\mu(E_n)}{2^n(1+\mu(E_n))} \le 1$. $\square$

## Duality of $L^p$ and $L^q$

**Def.** Let $F$ be a normed cplx vector space. A linear functional on $F$ is a function $\Phi : F \to \mathbb{C}$ satisfying $\Phi(f_1+f_2) = \Phi(f_1) + \Phi(f_2)$, $\Phi(\lambda f) = \lambda \cdot \Phi(f)$, $\forall f, f_1, f_2 \in F$ $\forall \lambda \in \mathbb{C}$.
We define the norm of $\Phi$ as $\|\Phi\| := \sup\{|\Phi(f)| : f \in F, \|f\|_F \le 1\}$.
If $\|\Phi\| < \infty$, then $\Phi$ is called a bounded linear functional on $F$.

Remarks: 1) By linearity, $|\Phi(\lambda f)| = |\lambda \Phi(f)| = |\lambda|\,|\Phi(f)|$, which implies that, equivalently, $\|\Phi\| = \sup\{|\Phi(f)| : \|f\|_F = 1\}$.

2) If $\|\Phi\| < \infty$, then $\|\Phi\| = \inf\{M \ge 0 : |\Phi(f)| \le M \cdot \|f\|_F \text{ for all } f \in F\}$.
In particular, $|\Phi(f)| \le \|\Phi\| \cdot \|f\|_F$, $\forall f \in F$.

**Prop.** For a linear functional $\Phi : F \to \mathbb{C}$, TFAE:
(i) $\Phi$ is bounded
(ii) $\Phi$ is continuous
(iii) $\Phi$ is continuous at some $f_0 \in F$.

Pf = Exercise.

**Def.** Given a normed cplx vector space $F$, the set $F^* := \{\Phi : F \to \mathbb{C} \mid \|\Phi\| < \infty\}$ of bdd linear functionals on $F$ is called the **dual** space of $F$.

**Prop.** If $F$ is a normed cplx vector space, then so is $F^*$ with the above defined norm.
In fact, $F^*$ is a complete metric space in this norm.

Pf = Exercise. /Hint: Completeness follows from that of $\mathbb{C}$./

(25)

Example: Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \le p < \infty$, and $q$ the conjugate of $p$. Then, by Hölder's Inequality, for any $g \in L^q(\mu)$, the map

$$\Phi_g : L^p(\mu) \to \mathbb{C}, \quad \Phi_g(f) := \int_X fg \, d\mu \quad \text{is in } (L^p(\mu))^*, \text{with } \|\Phi_g\| \le \|g\|_q.$$

**Thm.** Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space, $1 \le p < \infty$, and let $\Phi \in (L^p(\mu))^*$. Then, there exists a unique $g \in L^q(\mu)$, where $\frac{1}{q} + \frac{1}{p} = 1$, such that

(*) $\Phi(f) = \int_X fg \, d\mu$, for all $f \in L^p(\mu)$. Moreover, $\|\Phi\| = \|g\|_q$.

Pf. Uniqueness: Suppose $\Phi(f) = \int_X fg_1 \, d\mu = \int_X fg_2 \, d\mu$ for all $f$, for some $g_1, g_2 \in L^q(\mu)$. Then, for any $E \in \mathcal{M}$ with $\mu(E) < \infty$, taking $f := \chi_E$ gives $\int_E g_1 - g_2 \, d\mu = 0$, hence $g_1|_E = g_2|_E$ a.e. The $\sigma$-finiteness of $\mu$ then implies that $g_1 = g_2$ a.e. on $X$. ✓

Norm:

If $g$ satisfies (*), then by Hölder's Inequality, $\|\Phi\| \le \|g\|_q$.

Existence:

If $\|\Phi\| = 0$, then (*) is satisfied with $g \equiv 0$, so we may assume that $\|\Phi\| > 0$.

Suppose first that $\mu(X) < \infty$.

Define a function $\lambda : \mathcal{M} \to \mathbb{C}$ by $\lambda(E) := \Phi(\chi_E)$. We claim that $\lambda$ is a cplx measure. Indeed, $\lambda$ is finitely additive, since $\forall A, B \in \mathcal{M}$, $A \cap B = \emptyset$, $\chi_{A \cup B} = \chi_A + \chi_B$, so by linearity of $\Phi$, $\lambda(A \cup B) = \Phi(\chi_{A \cup B}) = \Phi(\chi_A) + \Phi(\chi_B) = \lambda(A) + \lambda(B)$. Now, if $E \in \mathcal{M}$ and $\{E_n\}_{n=1}^\infty \subseteq \mathcal{M}$ is a partition of $E$, then set $A_k := E_1 \cup \ldots \cup E_k$, for $k \in \mathbb{Z}_+$, and observe that $\|\chi_E - \chi_{A_k}\|_p = (\mu(E \setminus A_k))^{1/p} \xrightarrow[k \to \infty]{} 0$. Hence, by continuity of $\Phi$, $\lambda(A_k) \xrightarrow[k \to \infty]{} \lambda(E)$; i.e., $\lambda(E) = \lim_{k \to \infty} \sum_{n=1}^k \lambda(E_n) = \sum_{n=1}^\infty \lambda(E_n)$. ✓

We claim next that $\lambda \ll \mu$. Indeed, if $E \in \mathcal{M}$ is st. $\mu(E) = 0$, then $\|\chi_E\|_p = 0$, and hence $\Phi(\chi_E) = 0$, by linearity of $\Phi$. Thus, $\lambda(E) = 0$.

Therefore, by the Radon-Nikodym Thm., $\exists\, g \in L^1(\mu)$ st.

(**) $\Phi(\chi_E) = \lambda(E) = \int_E g \, d\mu = \int_X \chi_E g \, d\mu$, for every $E \in \mathcal{M}$

By linearity of $\Phi$, it follows that $\Phi(s) = \int_X sg \, d\mu$ for every simple function $s$, and hence $\Phi(f) \overset{(**)}{=} \int_X f g \, d\mu$, for every $f \in L^\infty(\mu)$, since every $f \in L^\infty(\mu)$ is a uniform limit of simple functions $(s_n)$. (Exercise!) But if $s_n \rightrightarrows f$, then $\|f - s_n\|_p \xrightarrow[n \to \infty]{} 0$ (b/c $\mu(X) < \infty$), and hence $\Phi(s_n) \xrightarrow[n \to \infty]{} \Phi(f)$.

We now want to show that $g \in L^q(\mu)$ and $\|g\|_q \le \|\Phi\|$.

Case 1: $p=1$.

By $(\overset{*}{*})$, for every $E \in \mathcal{M}$, $\left|\int_E g\,d\mu\right| \le |\Phi(\chi_E)| \le \|\Phi\| \cdot \|\chi_E\|_p \overset{p=1}{=} \|\Phi\|\,\|\chi_E\|_1 = \|\Phi\| \cdot \mu(E)$.

Thus, $|g(x)| \le \|\Phi\|$ for almost every $x \in X$, and hence $\|g\|_\infty \le \|\Phi\|$. ✓

Case 2: $p \in (1,\infty)$.

There is a measurable function $\alpha : X \to \mathbb{C}$ s.t. $|\alpha(x)|=1$, $\forall x$, and $\alpha \cdot g = |g|$. (Exercise)

For $n \in \mathbb{Z}_+$, let $E_n := \{x \in X : |g(x)| \le n\}$ and let $f_n := \chi_{E_n} \cdot |g|^{q-1}\alpha$.

Then, $\forall n$, $|f_n|^p \left(= |g|^{(q-1)p}\right) = |g|^q \left(= |g|^{q-1} \cdot |g|\right) = f_n \cdot g$ on $E_n$, and hence by $(\overset{*}{*}{*})$:

$$\int_{E_n} |g|^q\,d\mu = \int_X f_n g\,d\mu = \Phi(f_n) \le \|\Phi\| \cdot \|f_n\|_p = \|\Phi\| \cdot \left(\int_{E_n} (|g|^{q-1})^p d\mu\right)^{1/p} = \|\Phi\| \cdot \left(\int_{E_n} |g|^q d\mu\right)^{1/p}. \quad \Big| : \left(\int_{E_n} |g|^q d\mu\right)^{1/p}$$

Thus, $\left(\int_X \chi_{E_n} \cdot |g|^q d\mu\right)^{1/q} \le \|\Phi\|$, and so $\int_X \chi_{E_n} |g|^q d\mu \le \|\Phi\|^q$. By the M.C.T. applied to the increasing sequence $(\chi_{E_n}|g|^q)_{n=1}^\infty$, we get $\int_X |g|^q d\mu \le \|\Phi\|^q$; i.e., $g \in L^q(\mu)$ and $\|g\|_q \le \|\Phi\|$. ✓

Now, since $g \in L^q(\mu)$, it follows from Hölder's inequality that $\{f \mapsto \int_X fg\,d\mu\} \in (L^p(\mu))^*$. Therefore, $(\overset{*}{*}{*})$ means that $\Phi$ and $\int_X (\cdot) g\,d\mu$ are two continuous functions on $L^p(\mu)$ which coincide on $L^\infty(\mu)$.

b/c $\mu(X)<\infty \Rightarrow$ $f$ integrable iff $\sum_{n=1}^\infty \mu(\{x : f(x) \ge n\}) < \infty$, $f \ge 0$

But $L^\infty(\mu)$ is a dense subset of $L^p(\mu)$, so the functions must coincide on the whole $L^p(\mu)$.

This completes the proof if $\mu(X)<\infty$.

Suppose then that $\mu(X)=\infty$. Since $\mu$ is $\sigma$-finite, by the above lemma there is a function $w \in L^1(\mu)$ with $0<w<1$ on $X$. Then $\tilde{\mu} : \mathcal{M} \to [0,\infty)$ defined as $\tilde{\mu}(E) := \int_E w\,d\mu$ is a finite measure on $\mathcal{M}$, with the property that $\tilde{\mu}(E)=0 \Leftrightarrow \mu(E)=0$.

Since $w(x)>0$, $\forall x \in X$, it follows that the map $L^p(\tilde{\mu}) \ni f \mapsto w^{1/p} \cdot f \in L^p(\mu)$ is a linear isometry.

Therefore, $\Psi : L^p(\tilde{\mu}) \to \mathbb{C}$ defined as $\Psi(f) := \Phi(w^{1/p}f)$ is a bdd linear functional on $L^p(\tilde{\mu})$, with $\|\Psi\| = \|\Phi\|$.

By the first part of the proof, $\exists\, \tilde{g} \in L^q(\tilde{\mu})$ s.t. $\Psi(f) = \int_X f\tilde{g}\,d\tilde{\mu}$, and $\|\Phi\| = \|\tilde{g}\|_q$.

Put $g := w^{1/q}\tilde{g}$. (If $p=1$, then $g := \tilde{g}$.)

Then, for $p>1$, $\int_X |g|^q d\mu = \int_X |\tilde{g}|^q d\tilde{\mu} = \|\Psi\|^q = \|\Phi\|^q$, and for $p=1$, $\|g\|_\infty = \|\tilde{g}\|_\infty = \|\Psi\| = \|\Phi\|$,

and $\Phi(f) = \Psi(w^{-1/p}f) = \int_X w^{-1/p} f \tilde{g}\,d\tilde{\mu} = \int_X w^{1/q - 1} f\tilde{g}\,d\tilde{\mu} = \int_X f \cdot (w^{1/q}\tilde{g}) \cdot w^{-1} \cdot w\,d\mu = \int_X fg\,d\mu$, $\forall f \in L^p(\mu)$.

$\square$