Problem Set 2
due: February 9, 2026.

All numbered exercises are from the textbook *Real analysis for graduate students*, R.F. Bass, ver.3.1, 2016 (available through the course website).

1. Let X be a nonempty set. Let Δ be the collection of all outer measures on X, and let Λ be the collection of all pairs $(\mathcal{M}, \mu)$ such that $\mathcal{M}$ is a σ-algebra on X and μ is a measure on $\mathcal{M}$. For any $\alpha \in \Delta$, let $(\mathcal{M}_\alpha, \alpha_c) \in \Lambda$ denote the pair consisting of α-measurable sets $\mathcal{M}_\alpha$ and the measure $\alpha_c := \alpha|_{\mathcal{M}_\alpha}$. For $(\mathcal{M}, \mu) \in \Lambda$, let $\mu^0 \in \Delta$ denote the effect of Caratheodory construction on μ. Prove the following:

(a) If μ is complete and σ-finite, then $(\mu^0)_c = \mu$.

(b) For every $\mu \in \Lambda$, we have $((\mu^0)_c)^0 = \mu^0$.

(c) In general, it is not true that $((\alpha_c)^0)_c = \alpha_c$ (give an example).

[NB: Here, equality of measures is understood in the sense of functions; i.e., together with their σ-algebraic domains.]

2. (a) Show that every Vitali set $V \subset [0, 1]$ has positive Lebesgue outer measure.

(b) Show that for every $\epsilon > 0$, there exists a Vitali set V_ϵ satisfying $0 < m^*(V_\epsilon) < \epsilon$.

(c) Show that for every Lebesgue measurable set $A \subset V$, we have $m(A) = 0$.

3. Exercise 4.3.

4. Exercise 4.10. [Hint: Try proving the contrapositive. For that, assume first that $m(A) < \infty$ and use condition (ii) of the theorem on equivalent conditions for Lebesgue mesurability proved in class.]

5. Exercise 4.17.

6. (a) Let C be a subset of the $[0, 1]$ interval defined as $C = \bigcap_{n=1}^{\infty} C_n$, where $C_1 = [0, \frac{1}{5}] \cup [\frac{2}{5}, \frac{3}{5}] \cup [\frac{4}{5}, 1]$ and, for any $k \geq 1$, C_{k+1} is obtained from C_k by removing the second and fourth open fifths from each of the 3^k congruent closed intervals that C_k is composed of. Let m denote the Lebesgue measure in $\mathbb{R}$. Find $m(C)$ and the Hausdorff dimension of C. Justify your answers.

(b) Let C be a subset of the $[0, 1]$ interval defined as $C = \bigcap_{n=1}^{\infty} C_n$, where $C_1 = [0, \frac{2}{5}] \cup [\frac{3}{5}, 1]$ and, for any $k \geq 1$, C_{k+1} is obtained from C_k by removing the middle open fifth from each of the 2^k congruent closed intervals that C_k is composed of. Let m denote the Lebesgue measure in $\mathbb{R}$. Find $m(C)$ and the Hausdorff dimension of C. Justify your answers.

Practice Problems (not to be submitted):

7. Let $S \subset [0,1]^2 \subset \mathbb{R}^2$ denote the *Sierpiński carpet*; i.e., $S = \bigcap_{n=1}^{\infty} S_n$, where

$$S_1 = [0,1]^2 \setminus \left(\frac{1}{3}, \frac{2}{3}\right)^2$$

and, for any $k \geq 1$, S_{k+1} is obtained from S_k by removing the open middle ninth square from each of the 8^k congruent squares of area $\frac{1}{9^k}$ that S_k is composed of. Let m denote the Lebesgue measure in $\mathbb{R}^2$. Find $m(S)$ and the Hausdorff dimension of S. Justify your answers.

8. Using the terminology of Problem 1 above, prove the following:

(a) $(\alpha_c)^0 = \alpha$ iff α is regular.

(b) $(\mu^0)_c = \mu$ iff there exists a regular $\gamma \in \Delta$ such that $\mu = \gamma_c$.

9. (a) Show that every countable set $A \subset \mathbb{R}$ has Lebesgue measure zero.

(b) Give an example of a Lebesgue-measurable set $A \subset \mathbb{R}$ which contains no open interval and satisfies $m(A) > 0$. Justify.

10. Let m^* denote the Lebesgue outer measure in $\mathbb{R}^2$. Find $m^*(A)$ for the sets A from the following list. Which of the sets are m^*-measurable? Justify your answers.

(a) $A = \mathbb{Q} \times \mathbb{R}$.

(b) $A = C \times (\mathbb{R} \setminus \mathbb{Q})$, where $C \subset [0,1]$ is the ternary Cantor set.

(c) $A = \mathbb{Q} \times V$, where V is a Vitali set in $[0,1]$.

(d) $A = (\mathbb{R} \setminus \mathbb{Q}) \times V$, where V is a Vitali set in $[0,1]$.

11. Exercises 4.7, 4.8, 4.15, 4.16.