Problem Set 3
due: March 9, 2026.

All numbered exercises are from the textbook *Real analysis for graduate students*, R.F. Bass, ver.3.1, 2016 (available through the course website).

1. Let (X, d) be a metric space and let $f : X \to \mathbb{R}$. Define functions $m_f, M_f : X \to \overline{\mathbb{R}}$ as

$$m_f(x) = \sup\{\inf f(U) : U \text{ an open neighbourhood of } x\},$$
$$M_f(x) = \inf\{\sup f(U) : U \text{ an open neighbourhood of } x\}, \quad \text{for all } x \in X.$$

(a) Show that $m_f(x) \le f(x) \le M_f(x)$, for all $x \in X$.

(b) Prove that the functions m_f and M_f are Borel measurable. [Hint: Show that, for all $a \in \mathbb{R}$, the sets $\{x \in X : M_f(x) < a\}$ and $\{x \in X : m_f(x) > a\}$ are open.]

(c) Show that f is continuous at $x_0 \in X$ iff $m_f(x_0) = M_f(x_0)$.

2. For functions f and g on a measure space $(X, \mathcal{M}, \mu)$, we say that $f = g$ *almost everywhere* (*a.e.*, for short), when $\mu(\{x \in X : f(x) \ne g(x)\}) = 0$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $g : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function.

(a) Suppose that $f = g$ a.e. (with respect to Lebesgue measure). Prove that f is Lebesgue measurable.

(b) Suppose now that f is Lebesgue measurable. Prove that there exists a Borel measurable function $h : \mathbb{R} \to \mathbb{R}$ such that $f = h$ a.e. (with respect to Lebesgue measure).

3. Exercise 5.2.

4. Exercise 6.5.

5. Exercise 6.6.

6. Exercise 7.5.

7. Exercise 8.7.

Practice Problems (not to be submitted):

8. Exercises 5.3, 5.5, 5.8.

9. Exercises 6.2, 6.3, 6.7, 6.8.

10. Exercises 7.3, 7.7, 7.8, 7.9, 7.11–7.15, 7.25, 7.26

11. Exercises 8.5, 8.7, 8.8, 8.12.