Problem Set 4

due: April 6, 2026.

All numbered exercises are from the textbook *Real analysis for graduate students*, R.F. Bass, ver.3.1, 2016 (available through the course website).

1. Suppose $f:[0,1]^2\to\mathbb{R}$ is integrable with respect to the 2-dimensional Lebesgue measure m on $[0,1]^2$, and $\int_{[0,a]\times[0,b]} f\,dm = 0$ for all $a,b\in[0,1]$. Prove that $f=0$ a.e.

2. Let $(X,\mathcal{M},\mu)$ be a σ-finite measure space, and let $f:X\to\mathbb{R}$ be an $\mathcal{M}$-measurable function. Define the *distribution function* of f by

$$\mu_f(t) := \mu(\{x\in X : |f(x)|\geq t\}),\quad t>0.$$

(a) Show that $\mu_f:(0,\infty)\to[0,\mu(X)]$ is non-increasing and Borel measurable.

(b) Prove that, for any $p\in[1,\infty)$,

$$\int_X |f(x)|^p d\mu(x) = \int_0^\infty \mu_f(t)pt^{p-1}dt\,.$$

[Hint: $|f(x)|^p = \int_0^{|f(x)|} pt^{p-1}dt$.]

3. Let $n\geq 2$ and let S be a standard n-simplex in $\mathbb{R}^n$ with base of length a, for some $a>0$. That is,

$$S := \{(x_1,\ldots,x_n)\in\mathbb{R}^n : x_i\geq 0,\ \sum_{i=1}^n x_i\leq a\}\,.$$

Use Fubini Theorem (and induction) to find the Lebesgue integral $\int_{\mathbb{R}^n}\chi_S$.

4. Let $(X,\mathcal{M})$ be a measurable space.

(a) Prove that the collection of all complex measures on $(X,\mathcal{M})$ is a complex vector space (with addition and scalar multiplication defined as $(\mu+\lambda)(E) := \mu(E)+\lambda(E)$ and $(c\cdot\mu)(E) := c\cdot\mu(E)$, for $E\in\mathcal{M}$).

(b) Let $M(X)$ denote the complex vector space of all complex measures on $(X,\mathcal{M})$. Prove that the function defined as $\|\mu\| := |\mu|(X)$ is a norm on $M(X)$.

5. Suppose $\lambda,\lambda_1,\lambda_2$ are measures on a σ-algebra $\mathcal{M}$, and μ is a positive measure on $\mathcal{M}$. Prove the following statements:

(a) If λ is concentrated on a set $A\in\mathcal{M}$, then so is $|\lambda|$.

(b) If $\lambda_1\perp\lambda_2$, then $|\lambda_1|\perp|\lambda_2|$.

(c) If $\lambda_1\perp\mu$ and $\lambda_2\perp\mu$, then $(\lambda_1+\lambda_2)\perp\mu$.

(d) If $\lambda_1\ll\mu$ and $\lambda_2\ll\mu$, then $(\lambda_1+\lambda_2)\ll\mu$.

(e) If $\lambda\ll\mu$, then $|\lambda|\ll\mu$.

(f) If $\lambda_1\ll\mu$ and $\lambda_2\perp\mu$, then $\lambda_1\perp\lambda_2$.

(g) If $\lambda\ll\mu$ and $\lambda\perp\mu$, then $\lambda\equiv 0$.

6. Exercise 12.7.