

Practice Term Test 1

0. Practice problems from PS1 and PS2.
1. State the definitions of an algebra, σ -algebra, and monotone class of subsets of a set X .
 2. For each of the following statements, prove or give a specific counterexample (with justification):
 - (a) Every algebra \mathcal{A} on a set X is a σ -algebra.
 - (b) Every σ -algebra \mathcal{A} on a set X is a monotone class.
 - (c) Every monotone class \mathcal{M} on a set X is a σ -algebra.
 3. State the definitions of a measurable space, measure, and outer measure.
 4. Give an example of a finite set X and an outer measure μ^* on X which is not a measure.
 5. Let μ^* be an outer measure on a set X . State the definition of a μ^* -measurable set.
 6. (a) State the definition of a regular outer measure.
 (b) Let \mathcal{C} be a collection of subsets of a set X such that $\emptyset \in \mathcal{C}$, and let $\zeta : \mathcal{C} \rightarrow [0, \infty]$ be a function such that $\zeta(\emptyset) = 0$. State the Caratheodory construction of an outer measure μ^* from ζ .
 7. (a) Let (X, \mathcal{M}, μ) be a measure space. State the definition of a μ -null set.
 (b) Let $n \in \mathbb{Z}_+$ and let m (resp. m^*) denote the Lebesgue measure (resp. outer measure) on \mathbb{R}^n . Justify why every set $A \subset \mathbb{R}^n$ with $m^*(A) = 0$ is Lebesgue measurable and satisfies $m(A) = 0$.
 8. (a) State the definition of Borel measurable sets in a metric space (X, d) .
 (b) Let \mathcal{B} denote the σ -algebra of Borel measurable sets in \mathbb{R} and let m^* be the Lebesgue outer measure in \mathbb{R} . Prove or give a counterexample (with justification):
If $A \subset \mathbb{R}$ satisfies $m^(A) = 0$, then there exist $B, C \in \mathcal{B}$ such that $A = B \setminus C$.*
 9. Prove that for every Lebesgue measurable set $A \subset \mathbb{R}^n$, there exist Borel measurable sets $B, C \subset \mathbb{R}^n$, such that $C \subset A \subset B$ and $m(B \setminus C) = 0$.
 10. (a) Give an example (with justification) of a set X and a finite outer measure μ^* on X , subsets $A_n \uparrow A$ of X , and subsets $B_n \downarrow B$ of X such that $\mu^*(A_n)$ does not converge to $\mu^*(A)$ and $\mu^*(B_n)$ does not converge to $\mu^*(B)$.
 (b) Let (X, \mathcal{M}, μ) be a finite measure space, and let μ^* be the Caratheodory extension of μ . Show that if $A_n \uparrow A$ for subsets A_n, A of X , then $\mu^*(A) = \lim_{n \rightarrow \infty} \mu^*(A_n)$.
 11. Let (X, \mathcal{M}) be a measurable space.
 - (a) State the definition of a measurable function $f : X \rightarrow \overline{\mathbb{R}}$.
 - (b) Prove that $f : X \rightarrow \overline{\mathbb{R}}$ is measurable if and only if $f^{-1}(B) \in \mathcal{M}$ for every Borel measurable $B \subset \overline{\mathbb{R}}$.
 - (c) Give an example of (X, \mathcal{M}) and a non-measurable function $f : X \rightarrow \overline{\mathbb{R}}$ such that $f^{-1}(U) \in \mathcal{M}$ for every open set $U \subset \mathbb{R}$.
 - (d) Prove that $f : X \rightarrow \mathbb{R}$ is a simple function if and only if f is measurable and $f(X)$ is a finite set.
 12. Let $X = [0, 1]$ and let \mathcal{B} be the σ -algebra of Borel measurable subsets of X . Prove or give a counterexample (with justification):
For every Borel measurable function $f : (X, \mathcal{B}) \rightarrow (X, \mathcal{B})$ and every $A \in \mathcal{B}$, one has $f(A) \in \mathcal{B}$.
 13. (a) Prove that the Lebesgue integral of a non-negative simple function is well defined; i.e., independent of its representation as a non-negative combination of characteristic functions.
 (b) Prove that $\int(s_1 + s_2) = \int s_1 + \int s_2$, for any two non-negative simple functions s_1, s_2 .