Problem Set 1

due: February 1, 2019.

1. Let X be a non-empty finite set, and let $\mathcal{A}$ be a σ-algebra on X. Consider a relation on X:

$$x \sim y \iff [x \in A \Leftrightarrow y \in A, \text{ for all } A \in \mathcal{A}] .$$

(i) Show that the above is an equivalence relation on X.

(ii) Show that, for every $x \in X$, its equivalence class satisfies $[x]_\sim = \bigcap\{A \in \mathcal{A} : x \in A\}$ and $[x]_\sim \in \mathcal{A}$.

(iii) Let $E_1, \ldots, E_k$ be all the distinct equivalence classes in X modulo $\sim$. Show that $\mathcal{A}$ consists precisely of the empty set and unions of all sub-collections of $\{E_1, \ldots, E_k\}$ (i.e., $A \in \mathcal{A}$ iff $A = \varnothing$ or there exist $1 \le l \le k$ and $\{i_1, \ldots, i_l\} \subset \{1, \ldots, k\}$ such that $A = E_{i_1} \cup \cdots \cup E_{i_l}$).

2. Let $\mathcal{M} = \{A \subset \mathbb{R} : |A| \le \aleph_0 \text{ or } |A^c| \le \aleph_0\}$, and let $\mu : \mathcal{M} \to \{0, 1\}$ be a function defined as $\mu(A) = 0$ when $|A| \le \aleph_0$, and $\mu(A) = 1$ when $|A^c| \le \aleph_0$. Prove that $\mathcal{M}$ is a σ-algebra in $\mathbb{R}$ and μ is a measure on $\mathcal{M}$.

3. Let $X = \mathbb{R}$ and let $\mu^* : \mathcal{P}(X) \to \{0, \frac{1}{2}, 1\}$ be a function defined as $\mu^*(A) = 0$ when $|A| \le \aleph_0$, $\mu^*(A) = \frac{1}{2}$ when $|A| > \aleph_0$ and $|A^c| > \aleph_0$, and $\mu^*(A) = 1$ when $|A^c| \le \aleph_0$.

(a) Prove that μ^* is an outer measure on $\mathbb{R}$, which is not a measure.

(b) Find the σ-algebra of μ^*-measurable sets. Justify your answer.

4. Exercises 2.2, 2.5, 3.4, and 3.6 from the text.

5. Prove that every measure space admits a unique completion (Exercise 3.8).