Problem Set 2

due: February 15, 2019.

1. Let X be a set. Let Δ be the collection of all outer measures on X, and let Λ be the collection of all pairs $(\mathcal{M}, \mu)$ such that $\mathcal{M}$ is a σ-algebra on X and μ is a measure on $\mathcal{M}$. For any $\alpha \in \Delta$, let $(\mathcal{M}_\alpha, \alpha_c) \in \Lambda$ denote the pair consisting of α-measurable sets $\mathcal{M}_\alpha$ and the measure $\alpha_c := \alpha|_{\mathcal{M}_\alpha}$. For $(\mathcal{M}, \mu) \in \Lambda$, let $\mu^0 \in \Delta$ denote the effect of Caratheodory construction on μ. Prove the following:

(a) $(\alpha_c)^0 = \alpha$ iff α is regular.

(b) $(\mu^0)_c = \mu$ iff there exists a regular $\gamma \in \Delta$ such that $\mu = \gamma_c$.

(c) If μ is complete and σ-finite, then $(\mu^0)_c = \mu$.

(d) For every $\mu \in \Lambda$, we have $((\mu^0)_c)^0 = \mu^0$.

[NB: Here, equality of measures is understood in the sense of functions; i.e., together with their σ-algebraic domains.]

2. Exercises 4.6 and 4.7 from the text.

3. (a) Show that every Vitali set $V \subset [0,1]$ has positive Lebesgue outer measure.

(b) Prove that for every $\epsilon > 0$, there exists a Vitali set V_ϵ satisfying $0 < m^*(V_\epsilon) < \epsilon$.

(c) Prove that for every Lebesgue measurable set $A \subset V$, we have $m(A) = 0$.

4. Let $C \subset [0,1]$ denote the ternary Cantor set (i.e., $C = \bigcap_{n\in\mathbb{N}} C_n$, where $C_0 = [0,1]$, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, and C_{n+1} is obtained from C_n by removing the open middle thirds from all the closed intervals whose union constitutes C_n.) Find the Lebesgue measure of C. Justify your answer.

5. Prove that if $A \subset \mathbb{R}$ is Lebesgue measurable and $m(A) > 0$, then A contains a Lebesgue non-measurable set.

6. Exercise 4.17 from the text.

7. Recall that, for $p \in [0, \infty)$, the *p-dimensional Hausdorff outer measure* on a metric space (X, d) is defined by

$$h^{p*}(Y) = \sup_{\delta>0} \inf \left\{ \sum_{k=1}^{\infty} h^p(Y_k) \mid \{Y_k\}_{k=1}^{\infty} \subset \mathcal{P}(X), Y \subset \bigcup_k Y_k, \operatorname{diam}(Y_k) < \delta, \forall k \right\},$$

for $Y \subset X$, where $h^p(Z) = \frac{\alpha(p)}{2^p} \operatorname{diam}(Z)^p$ for $Z \neq \varnothing$, $h^p(\varnothing) = 0$, and $\alpha(p) = \frac{(\Gamma(\frac{1}{2}))^p}{\Gamma(\frac{p}{2}+1)}$.

For $\delta > 0$, denote by h^p_δ the function on $\mathcal{P}(X)$ defined by

$$h^p_\delta(Y) = \inf \left\{ \sum_{k=1}^{\infty} h^p(Y_k) \mid \{Y_k\}_{k=1}^{\infty} \subset \mathcal{P}(X), Y \subset \bigcup_k Y_k, \operatorname{diam}(Y_k) < \delta, \forall k \right\}.$$

By the Caratheodory Extension Theorem, h^p_δ is an outer measure on X.
Prove that, for every $Y \subset X$, the function $(0, \infty) \ni \delta \mapsto h^p_\delta(Y) \in [0, \infty]$ is decreasing. Prove that h^{p*} is an outer measure on X.