Practice Term Test 1

1. State the definitions of an algebra, σ-algebra, and monotone class of subsets of a set X.

2. For each of the following, prove or give a specific counterexample (with justification):

(a) Every algebra $\mathcal{A}$ on a set X is a σ-algebra.
(b) Every σ-algebra $\mathcal{A}$ on a set X is a monotone class.
(c) Every monotone class $\mathcal{M}$ on a set X is a σ-algebra.
(d) If $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$ are σ-algebras on X, then $\bigcup_i \mathcal{A}_i$ is a σ-algebra on X.
(e) If $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots$ are monotone classes on X, then $\bigcup_i \mathcal{M}_i$ is a monotone class on X.

3. State the definitions of a measurable space, measure, and outer measure.

4. Give an example of a finite set X and an outer measure μ^* on X which is not a measure.

5. Let μ^* be an outer measure on a set X. State the definition of a μ^*-measurable set.

6. (a) State the definition of a regular outer measure.
(b) Let $\mathcal{C}$ be a collection of subsets of a set X such that $\varnothing \in \mathcal{C}$, and let $\zeta : \mathcal{C} \to [0, \infty]$ be a function such that $\zeta(\varnothing) = 0$. State the Caratheodory construction of an outer measure μ^* from ζ.
(c) Prove that the μ^* resulting from Caratheodory's construction is regular.

7. (a) Let $(X, \mathcal{M}, \mu)$ be a measure space. State the definition of a μ-null set.
(b) Let $n \in \mathbb{Z}_+$ and let m (resp. m^*) denote the Lebesgue measure (resp. outer measure) on $\mathbb{R}^n$. Prove that every set $A \subset \mathbb{R}^n$ with $m^*(A) = 0$ is Lebesgue measurable and satisfies $m(A) = 0$.

8. (a) State the definition of Borel measurable sets in a metric space (X, d).
(b) Let $\mathcal{B}$ denote the σ-algebra of Borel measurable sets in $\mathbb{R}$ and let m^* be the Lebesgue outer measure in $\mathbb{R}$. Prove or give a counterexample (with justification): If $A \subset \mathbb{R}$ satisfies $m^*(A) = 0$, then there exist $B, C \in \mathcal{B}$ such that $A = B \setminus C$.

9. Let m^* denote the Lebesgue outer measure in $\mathbb{R}^2$. Find $m^*(A)$ for the sets A from the following list. Which of the sets are m^*-measurable? Justify your answers.

(a) $A = \{0\} \times \mathbb{R}$.
(b) $A = \mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})$.
(c) $A = \mathbb{Q} \times \mathbb{R}$.
(d) $A = \mathbb{Q} \times V$, where V is a Vitali set in $[0, 1]$.

10. Exercises 5.1, 5.3, 5.8, from the text.

11. Let $(X, \mathcal{M})$ be a measurable space and let $f : X \to \mathbb{R}$.

(a) Prove that f is a measurable function if and only if $f^{-1}(B) \in \mathcal{M}$ for every Borel measurable $B \subset \mathbb{R}$.
(b) Prove that f is a simple function if and only if $f(X)$ is a finite set and f is measurable.

12. Let $S \subset [0, 1]^2 \subset \mathbb{R}^2$ denote the *Sierpiński carpet*; i.e., $S = \bigcap_{i=1}^{\infty} S_i$, where

$$S_1 = [0,1]^2 \setminus \left(\frac{1}{3}, \frac{2}{3}\right)^2,$$

and S_{i+1} is obtained from S_i by removing the open middle ninth square from each of the 8^i congruent squares of area $\frac{1}{9^i}$ that S_i is composed of.
Let m denote the Lebesgue measure in $\mathbb{R}^2$. Find $m(S)$. Find $\dim_{\mathcal{H}}(S)$. Justify your answers.