MEASURE THEORY LECTURE NOTES

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XI. Complex Measures

Definition 1. Let $\mathcal{M}$ be a σ-algebra on a set X. A *complex measure* on $\mathcal{M}$ is a complex-valued function $\mu : \mathcal{M} \to \mathbb{C}$ satisfying

$$\mu(\bigcup_n E_n) = \sum_n \mu(E_n)\,, \tag{1}$$

for every denumerable collection $\{E_n\}_n$ of pairwise disjoint elements of $\mathcal{M}$.

Remark 2.
1. It follows from the definition that $\mu(\varnothing) = 0$. Indeed, we have $\varnothing = \varnothing \sqcup \varnothing$ and hence $\mu(\varnothing) = 2\mu(\varnothing)$, which in light of $\mu(\varnothing) \in \mathbb{C}$ implies $\mu(\varnothing) = 0$.
2. Since the left side of (1) is a complex number, it follows that the series on the right side of (1) is always convergent. Moreover, the sum of the series is independent of the ordering of the sets E_n, and hence the series is *absulutely* convergent.

Example 3. The following is a model example of a complex measure. By the Radon-Nikodym Theorem below, all complex measures arise in this way modulo a set of measure zero.

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let $h : X \to \mathbb{C}$ be an integrable function. (Recall that a complex-valued function $f = u + iv$, where u and v are real-valued, is called integrable when both u and v are integrable as real-valued functions. In this case, one defines $\int f := \int u + i \int v$.) Then, the function $\nu : \mathcal{M} \to \mathbb{C}$ defined as

$$\nu(A) \;:=\; \int_A h\,d\mu \tag{2}$$

is a complex measure. The proof is an elementary exercise.

Definition 4. Let $\mu : \mathcal{M} \to \mathbb{C}$ be a complex measure on a σ-algebra $\mathcal{M}$. The function $|\mu| : \mathcal{M} \to \mathbb{R}$ defined as

$$|\mu|(E) := \sup\{\sum_{n=1}^{\infty} |\mu(E_n)| : \{E_n\}_n \subset \mathcal{M}, E = \bigcup_{n=1}^{\infty} E_n, E_j \cap E_k = \varnothing \text{ for } j \neq k\}$$

is called the *total variation measure of* μ.

Remark 5. Observe that we always have $|\mu|(A) \geq |\mu(A)|$, for every $A \in \mathcal{M}$. Indeed, the family $\{A, \varnothing, \varnothing, \ldots\}$ forms a measurable partition of A, and hence

$$|\mu|(A) \geq |\mu(A)| + \sum_{n=2}^{\infty} |\mu(\varnothing)| = |\mu(A)|\,.$$

Exercise 6. For a signed measure μ on a σ-algebra $\mathcal{M}$, one defines the *total variation measure of* μ as $|\mu| := \mu^+ + \mu^-$, where μ^+ and μ^- are the unique positive measures from the Jordan Decomposition Theorem for μ. Note that a *real-valued* (i.e., finite) signed measure μ may be regarded as a complex measure. Prove that if μ is a real-valued signed measure, then the two definitions of $|\mu|$ coincide.

Theorem 7. *The total variation $|\mu|$ of a complex measure μ on a σ-algebra $\mathcal{M}$ is a positive measure on $\mathcal{M}$.*

Proof. Let $\{E_n\}_{n=1}^\infty \subset \mathcal{M}$ be an arbitrary collection of pairwise disjoint measurable sets. Set $E := \bigcup_n E_n$. We want to show that

$$|\mu|(E) = \sum_{n=1}^\infty |\mu|(E_n)\,.$$

To this end, for every $n \in \mathbb{Z}_+$, choose an arbitrary real number t_n satisfying $|\mu|(E_n) > t_n$. Then, by definition of $|\mu|$, there exists for every n a measurable partition $\{A_{nk}\}_{k=1}^\infty$ of E_n such that $\sum_{k=1}^\infty |\mu(A_{nk})| \geq t_n$. Since the (countable) family $\{A_{nk}\}_{n,k=1}^\infty$ forms a partition of E, we get that

$$|\mu|(E) \geq \sum_{n,k=1}^\infty |\mu(A_{nk})| = \sum_{n=1}^\infty \left(\sum_{k=1}^\infty |\mu(A_{nk})| \right) \geq \sum_{n=1}^\infty t_n\,.$$

(Note that above we used the absolute convergence of the series in question to rearrange its terms without altering the sum.)
Taking supremum over all sequences $(t_n)_{n=1}^\infty$ as above, we get

$$|\mu|(E) \geq \sum_{n=1}^\infty |\mu|(E_n)\,.$$

For the proof of the opposite inequality, let $\{A_k\}_{k=1}^\infty \subset \mathcal{M}$ be another arbitrary partition of E. Then, for every k, $\{A_k \cap E_n\}_{n=1}^\infty$ is a measurable partition of A_k, and for every n, $\{A_k \cap E_n\}_{k=1}^\infty$ is a measurable partition of E_n. Thus, again by absolute convergence, we get

$$\begin{aligned}\sum_{k=1}^\infty |\mu(A_k)| = \sum_{k=1}^\infty \left| \sum_{n=1}^\infty \mu(A_k \cap E_n) \right| &\leq \sum_{k=1}^\infty \left(\sum_{n=1}^\infty |\mu(A_k \cap E_n)| \right) \\ &= \sum_{n=1}^\infty \left(\sum_{k=1}^\infty |\mu(A_k \cap E_n)| \right) \leq \sum_{n=1}^\infty |\mu|(E_n)\,.\end{aligned}$$

Taking supremum over all countable measurable partitions $\{A_k\}_k$ of E, we get $|\mu|(E) \leq \sum_{n=1}^\infty |\mu|(E_n)$. This proves countable additivity of $|\mu|$.

The equality $|\mu|(\varnothing) = 0$ follows from $\mu(\varnothing) = 0$ and the fact that the only countable measurable partition of $\varnothing$ consists of copies of $\varnothing$. □

Perhaps even more interestingly, the total variation of any complex measure is a *finite* positive measure, as the following theorem shows.

Theorem 8. *If μ is a complex measure on a measurable space $(X, \mathcal{M})$, then the total variation measure $|\mu|$ satisfies $|\mu|(X) < +\infty$.*

We shall first establish an auxiliary lemma, which is a somewhat surprising complex analysis result of independent interest.

Lemma 9. *Given any collection $\{z_1, \ldots, z_N\}$ of (not necessarily pairwise distinct) N complex numbers, there exists an index subset $S \subset \{1, \ldots, N\}$ such that*

$$\left|\sum_{n\in S} z_n\right| \;\geq\; \frac{1}{\pi}\cdot\sum_{n=1}^{N}|z_n|\,.$$

Proof. For $1 \leq n \leq N$, let $\alpha_n \in (-\pi, \pi]$ be such that $z_n = |z_n|e^{i\alpha_n}$. For $\vartheta \in [-\pi, \pi]$, let

$$S(\vartheta) \coloneqq \{n \in \{1, \ldots, N\} : \cos(\alpha_n - \vartheta) > 0\}\,.$$

Then,

$$\begin{aligned}|\sum_{n\in S(\vartheta)} z_n| = |e^{-i\vartheta}|\cdot|\sum_{n\in S(\vartheta)} z_n| = |\sum_{n\in S(\vartheta)} e^{-i\vartheta}z_n| \geq \mathrm{Re}\left(\sum_{n\in S(\vartheta)}|z_n|e^{i(\alpha_n-\vartheta)}\right)\\ = \sum_{n\in S(\vartheta)}|z_n|\cos(\alpha_n-\vartheta) = \sum_{n=1}^{N}|z_n|\cos^+(\alpha_n-\vartheta)\,,\end{aligned}$$

where, as usual, $\cos^+ = \max\{\cos, 0\}$.

Now, choose $\vartheta_0 \in [-\pi, \pi]$ so as to maximize the latter sum, and set $S \coloneqq S(\vartheta_0)$. By the Mean Value Theorem for Riemann integral, we have

$$\sum_{n=1}^{N}|z_n|\cos^+(\alpha_n-\vartheta_0) \geq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=1}^{N}|z_n|\cos^+(\alpha_n-\vartheta)\right)d\vartheta\,,$$

and hence

$$\begin{aligned}|\sum_{n\in S} z_n| &\geq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=1}^{N}|z_n|\cos^+(\alpha_n-\vartheta)\right)d\vartheta\\ &= \sum_{n=1}^{N}\left(|z_n|\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^+(\alpha_n-\vartheta)\,d\vartheta\right) = \sum_{n=1}^{N}(|z_n|\cdot\frac{1}{\pi}) = \frac{1}{\pi}\cdot\sum_{n=1}^{N}|z_n|\,,\end{aligned}$$

where the penultimate equality follows from the fact that $\cos^+$ is periodic with period 2π, and hence $\int_{-\pi}^{\pi}\cos^+(\alpha_n-\vartheta)\,d\vartheta = 2$ independently of the choice of α_n. □

Proof of Theorem 8. Suppose first that, for some $E \in \mathcal{M}$, we have $|\mu|(E) = +\infty$. Set $t \coloneqq \pi(1 + |\mu(E)|)$. (Of course, $t < +\infty$, since μ is a complex measure.)

Since $|\mu|(E) = +\infty$, then by definition of $|\mu|$ there exist a measurable partition $\{E_n\}_{n=1}^{\infty}$ of E and a positive integer N such that

$$\sum_{n=1}^{N}|\mu(E_n)| > t\,.$$

Applying Lemma 9 with $z_n \coloneqq \mu(E_n)$, we conclude that there is a measurable set $A \subset E$ (namely, the union of some of the $E_1, \ldots, E_N$) such that

$$|\mu(A)| > \frac{t}{\pi} \geq 1\,.$$

Moreover, for $B \coloneqq E \setminus A$, we also have

$$|\mu(B)| = |\mu(E\setminus A)| = |\mu(E)-\mu(A)| \geq |\mu(A)| - |\mu(E)| > \frac{t}{\pi} - |\mu(E)| = 1\,.$$

We thus have $E = A \sqcup B$, with $|\mu(A)| > 1$ and $|\mu(B)| > 1$. By additivity of $|\mu|$, at least one of the A, B must be of infinite $|\mu|$-measure.

Now, if $|\mu|(X) = +\infty$, we construct recursively an infinite sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{M}$ of pairwise disjoint sets with $|\mu(A_n)| > 1$ for all n, as follows: By the first part of the proof, we can partition X into A_1 and B_1 such that $|\mu(A_1)| > 1$ and $|\mu|(B_1) = +\infty$. Having defined $A_1, \dots, A_k$ and $B_1, \dots, B_k$, partition B_k into A_{k+1} and B_{k+1} such that $|\mu(A_{k+1})| > 1$ and $|\mu|(B_{k+1}) = +\infty$.

Then, by additivity of μ, we have $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$. In particular, the series $\sum_{n=1}^{\infty} \mu(A_n)$ is absolutely convergent. Hence, by the Basic Divergence Test, $\lim_{n\to\infty} |\mu(A_n)| = 0$, contradicting the choice of the A_n. □

Exercise 10. (a) Show that if μ, ν are complex measures on a σ-algebra $\mathcal{M}$, then so are $\mu + \nu$ and $c \cdot \mu$ for any $c \in \mathbb{C}$ (where $(\mu+\nu)(E) = \mu(E) + \nu(E)$ and $(c\mu)(E) = c \cdot \mu(E)$ for $E \in \mathcal{M}$). Therefore the set of all complex measures on $\mathcal{M}$ forms a complex vector space.

(b) Prove that the function $\|\mu\| \coloneqq |\mu|(X)$ defines a norm on that vector space.

Absolute Continuity.

Definition 11. Let μ be a positive measure on a σ-algebra $\mathcal{M}$, and let λ be an arbitrary measure on $\mathcal{M}$. We say that λ is *absolutely continuous* with respect to μ, and write $\lambda \ll \mu$, when $\lambda(E) = 0$ for all $E \in \mathcal{M}$ with $\mu(E) = 0$.

For the following proposition, recall that a (positive, signed, or complex) measure μ is said to be concentrated on a measurable set A, when $\mu(E) = 0$ for every measurable $E \subset A^c$.

Proposition 12. *Suppose $\lambda, \lambda_1, \lambda_2$ are arbitrary measures on a σ-algebra $\mathcal{M}$, and μ is a positive measure on $\mathcal{M}$. Then:*

(i) *If λ is concentrated on $A \in \mathcal{M}$, then so is $|\lambda|$.*
(ii) *If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$.*
(iii) *If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $(\lambda_1 + \lambda_2) \perp \mu$.*
(iv) *If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $(\lambda_1 + \lambda_2) \ll \mu$.*
(v) *If $\lambda \ll \mu$, then $|\lambda| \ll \mu$.*
(vi) *If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.*
(vii) *If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda \equiv 0$.*

Proof. (i) Supposed first that λ is a signed measure. By the Hahn and Jordan decomposition theorems, there exist a set $E \in \mathcal{M}$ and positive measures λ^+, λ^- on $\mathcal{M}$ all such that $\lambda = \lambda^+ - \lambda^-$, λ^+ is concentrated on E^c, and λ^- is concentrated on E. Let then $B \in \mathcal{M} \cap \mathcal{P}(A^c)$ be arbitrary. We have

$$\begin{aligned}\lambda^+(B) &= \lambda^+(B \cap E) + \lambda^+(B \cap E^c) = 0 + \lambda^+(B \cap E^c) \\ &= -\lambda^-(B \cap E^c) + \lambda^+(B \cap E^c) = \lambda(B \cap E^c) = 0\,,\end{aligned}$$

since $B \cap E \subset E$ and $B \cap E^c \subset E^c \cap A^c$. Similarly,

$$\begin{aligned}\lambda^-(B) &= \lambda^-(B \cap E) + \lambda^-(B \cap E^c) = \lambda^-(B \cap E) + 0 \\ &= \lambda^-(B \cap E) + (-\lambda^+(B \cap E)) = -\lambda(B \cap E) = 0\,.\end{aligned}$$

Thus, $|\lambda|(B) = \lambda^+(B) + \lambda^-(B) = 0$.

Now, suppose λ is a complex measure. Let $B \in \mathcal{M} \cap \mathcal{P}(A^c)$ be arbitrary, and let $\{B_n\}_{n=1}^\infty$ be an arbitrary measurable partition of B. Then, for each n, $B_n \subset A^c$ and hence $\lambda(B_n) = 0$. Consequently, $\sum_{n=1}^\infty |\lambda(B_n)| = 0$. Since the $\{B_n\}$ were arbitrary, we get $|\lambda|(B) = 0$.

Property (ii) follows directly from (i) and definition of 'mutually singular'.

For the proof of (iii), choose $A_1, B_1, A_2, B_2 \in \mathcal{M}$ such that $B_1 = A_1^c$, $B_2 = A_2^c$, λ_1 is concentrated on A_1, λ_2 is concentrated on A_2, and μ is concentrated on B_1 as well as on B_2. The latter property means that μ is concentrated on $B_1 \cap B_2$. Indeed, for any measurable $E \subset (B_1 \cap B_2)^c$, we have $E = E_1 \cup E_2$ with $E_i = E \cap B_i^c$, and hence $0 \leq \mu(E) \leq \mu(E_1) + \mu(E_2) = 0 + 0$. Since $\lambda_1 + \lambda_2$ is concentrated on $A_1 \cup A_2$, the result follows.

Property (iv) is trivial.

For (v), suppose first that λ is a signed measure, and let $E \in \mathcal{M}$ and λ^+, λ^- be as in the proof of (i). Let $B \in \mathcal{M}$ be such that $\mu(B) = 0$. Then, by monotonicity of μ, $\mu(B \cap E) = 0 = \mu(B \cap E^c)$ as well. Since $\lambda \ll \mu$, we get

$$\begin{aligned}\lambda^+(B) &= \lambda^+(B \cap E) + \lambda^+(B \cap E^c) = 0 + \lambda^+(B \cap E^c) \\ &= -\lambda^-(B \cap E^c) + \lambda^+(B \cap E^c) = \lambda(B \cap E^c) = 0\,,\end{aligned}$$

and

$$\begin{aligned}\lambda^-(B) &= \lambda^-(B \cap E) + \lambda^-(B \cap E^c) = \lambda^-(B \cap E) + 0 \\ &= \lambda^-(B \cap E) + (-\lambda^+(B \cap E)) = -\lambda(B \cap E) = 0\,,\end{aligned}$$

and hence $|\lambda|(B) = \lambda^+(B) + \lambda^-(B) = 0$.

Now, suppose λ is a complex measure. Let $B \in \mathcal{M}$ be an arbitrary set with $\mu(B) = 0$, and let $\{B_n\}_{n=1}^\infty$ be an arbitrary measurable partition of B. Then, for each n, $B_n \subset B$, hence $\mu(B_n) = 0$ and so $\lambda(B_n) = 0$. Consequently, $\sum_{n=1}^\infty |\lambda(B_n)| = 0$. Since the $\{B_n\}$ were arbitrary, we get $|\lambda|(B) = 0$.

Proofs of properties (vi) and (vii) are left as an exercise. □

We finish this section with a statement of a complex-measure version of the Radon-Nikodym theorem. The proof will be covered in an in-class presentation.

Theorem 13 (Radon-Nikodym)**.** *Let μ be a positive σ-finite measure on a σ-algebra $\mathcal{M}$ on X, and let λ be a complex measure on $\mathcal{M}$. Then:*

(i) *There is a unique pair (λ_a, λ_s) of complex measures on $\mathcal{M}$ such that*

$$\lambda = \lambda_a + \lambda_s, \quad \lambda_a \ll \mu, \quad \lambda_s \perp \mu.$$

Moreover, if λ is positive finite, then so are λ_a and λ_s.

(ii) *There is a unique (a.e.) integrable function $h : X \to \mathbb{C}$ such that*

$$\lambda(E) = \int_E h\, d\mu \qquad \text{for all } E \in \mathcal{M}.$$

Definition 14. The pair (λ_a, λ_s) is called the *Lebesgue decomposition* of λ relative to μ.

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