Practice Term Test 2

1. Exercises 7.4, 7.6, 7.10, 7.12, 7.13, 7.15, 7.26.

2. Exercises 8.1, 8.3, 8.5, 8.7, 8.8.

3. Exercises 11.3, 11.4, 11.5, 11.10, 11.11, 11.12, 11.15.

4. Let $n \geq 2$ and let S be a standard n-simplex in $\mathbb{R}^n$ with base of length a, for some $a > 0$. That is,

$$S := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ \sum_{i=1}^{n} x_i \leq a\}\,.$$

Use Fubini Theorem (and induction) to find the Lebesgue integral $\displaystyle\int_{\mathbb{R}^n} \chi_S$.

For Problems 5 and 6, let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let $f : X \to \mathbb{R}$ be an $\mathcal{M}$-measurable function. Define the *distribution function* of f by

$$\mu_f(t) := \mu(\{x \in X : |f(x)| \geq t\}), \quad t > 0.$$

5. Show that $\mu_f : (0, \infty) \to [0, \mu(X)]$ is non-increasing and Borel measurable.

6. Prove that, for any $p \in [1, \infty)$,

$$\int_X |f(x)|^p d\mu(x) = \int_0^\infty \mu_f(t) p t^{p-1} dt\,.$$

Hint: $|f(x)|^p = \int_0^{|f(x)|} p t^{p-1} dt$.