Problem Set 1

January 24, 2025.

0. Review (and comprehend) the proof of the convergent version of Hironaka Division Theorem (3.4–3.6). [Do NOT submit!]

1. Let $f : M \to N$ be a holomorphic mapping of (non-empty) complex manifolds. Show that the definition of rank of f at a point $\xi \in M$ (Def. 1.9 in Lecture Notes) is independent of the choice of coordinate charts at ξ and $f(\xi)$. Prove that, for any $k \in \mathbb{N}$, the set $\{x \in M : \mathrm{rk}_x f \le k\}$ is analytic in M.

2. Prove the converse of Thm. 2.13: If M is an m-dimensional manifold and $X \subset M$ is analytic of pure dimension $m-1$, then X is locally principal.

[Hint: Suppose that in a neighbourhood U of a point $\xi \in X$ we have $X \cap U = h_1^{-1}(0) \cap \cdots \cap h_s^{-1}(0)$ for some $h_1, \dots, h_s \in \mathcal{O}(U)$, and let $Z := h_1^{-1}(0)$. By Thm. 2.11, reg$Z$ is an open dense subset of Z, and hence an $(m-1)$-dimensional manifold, by Thm. 2.13. Consider a connected component $\tilde{Z}$ of reg$Z$. For $j = 2, \dots, s$, we have either $\tilde{Z} \subset h_j^{-1}(0)$ or else $\tilde{Z} \cap h_j^{-1}(0)$ is a nowhere-dense subset of $\tilde{Z}$ (Thm. 2.7), hence of dimension strictly less than $m-1$.]

3. (a) Let $U \subset \mathbb{C}^n$ be a non-empty connected open set, $n \ge 2$, and let X be the zero set of a single holomorphic function $h \in \mathcal{O}(U)$. Show that $a \in \mathrm{sng}X$ only if

$$h(a) = \frac{\partial h}{\partial x_1}(a) = \cdots = \frac{\partial h}{\partial x_n}(a) = 0\,.$$

(b) Show that if X is an analytic subset of a 2-dimensional manifold M, then sng$X$ is an analytic subset of M.

4. Let $X = \{(x,y,z) \in \mathbb{C}^3 : x^2 - yz = x - xz = 0\}$. Show that X is the union of three irreducible (globally) analytic sets in $\mathbb{C}^3$. Find the singular loci of X and each of its three irreducible components.

5. Consider the total ordering of $\mathbb{N}^3$ by lexicographic ordering of 4-tuples $(|\beta|, \beta_x, \beta_y, \beta_z)$, where $\beta = (\beta_x, \beta_y, \beta_z) \in \mathbb{N}^3$. Find the (vertices of the) diagram $\mathfrak{N}(I)$ of initial exponents of the ideal I in $\mathbb{C}\{x,y,z\}$, where

(a) $I = (x,y)^3$

(b) $I = (x^2, z - y^2 e^x, y^3 - xz)$

(c) $I = (x,y,z)^k$, $k \in \mathbb{Z}_+$

(d) $I = (x - yz, y^2 - xz, z^3 - xy)\,.$

Justify your answers.

6. Formulate (and prove) a criterion for finiteness of $\dim_{\mathbb{C}} \mathbb{C}\{x_1, \dots, x_n\}/I$ in terms of the vertices of the diagram $\mathfrak{N}(I)$.

7. Find the standard bases of the ideals of Problem 5.