

Problem Set 2

February 9, 2025.

1. Prove Remarks 4.17, 4.18, and 4.19 of the Lecture Notes.
2. Prove Lemmas 5.1 and 5.4.
3. Let A be a UFD, and let B be an integral domain, integral over A . Prove that, for every $b \in B$, there is a unique monic polynomial $P \in A[w]$ of minimal degree, such that $P(b) = 0$.
4. Let X be an analytic subset of a manifold M , and let $\xi \in X$. We define the *local ring of X at ξ* as

$$\mathcal{O}_{X,\xi} := \mathcal{O}_{M,\xi} / \mathfrak{I}(X_\xi),$$

where $\mathfrak{I}(X_\xi)$ denotes the full ideal of the analytic germ X_ξ . Prove the following:

- (a) The ring $\mathcal{O}_{X,\xi}$ is an integral domain if and only if X_ξ is an irreducible germ.
 - (b) The ring $\mathcal{O}_{X,\xi}$ need not be a UFD, even if X is locally principal and locally irreducible (give an example and justify).
5. Recall that the *Krull dimension* of a Noetherian local ring (R, \mathfrak{m}) is the maximal length d of a chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d = \mathfrak{m}$ in R . Let X be an analytic subset of a manifold M , and let $\xi \in X$.
 - (a) Prove that the zero ideal (0) is prime in $\mathcal{O}_{X,\xi}$ if and only if X_ξ is an irreducible germ.
 - (b) Suppose that X_ξ is irreducible. Prove that the Krull dimension of $\mathcal{O}_{X,\xi}$ equals $\dim_\xi X$.
 6. Recall that a Noetherian local ring (R, \mathfrak{m}) is called *regular* when the maximal ideal \mathfrak{m} can be generated by n elements, where n is the Krull dimension of R . Show that, if X_ξ is smooth, then the local ring $\mathcal{O}_{X,\xi}$ is regular.