Problem Set 3
March 9, 2025.

1. Let X be a k-dimensional analytic subset of a manifold M, and let, for every $d \in \mathbb{N}$,

$$X^{(d)} = \{x \in X : \dim_x X = d\}.$$

Prove the following:

(a) $X^{(k)}$ is an analytic subset of X (in particular, closed!).

(b) For every $d \in \mathbb{N}$, $\overline{X^{(d)}}$ is analytic of pure dimension d. Moreover,

$$X = X^{(k)} \cup X^{(k-1)} \cup \cdots \cup X^{(0)} = X^{(k)} \cup \overline{X^{(k-1)}} \cup \cdots \cup \overline{X^{(0)}}.$$

(c) For every $d = 1, \dots, k$, $\bigcup_{i<d} X^{(i)}$ is open in X.

(d) For every $d = 1, \dots, k$, $X \setminus \bigcup_{i<d} X^{(i)}$ is analytic in X.

2. Let X and $X^{(d)}$ be as above. Give, as explicit as possible, a characterization of $X^{(d)}$ ($d = 0, \dots, k$) in terms of components of $\mathrm{Reg}(X)$.

3. Prove Proposition 8.8 and Remark 8.13(1) from Lecture Notes.

4. (a) [Hartshorne, Ex.II.1.2(c)] Show that a sequence of sheaves on a topological space X is exact iff, for every $\xi \in X$, the corresponding sequence of stalks at ξ is exact.

(b) Use part (a) to conclude that a morphism of sheaves $\alpha : \mathcal{F} \to \mathcal{G}$ is injective (resp. surjective) iff α_ξ is so for every $\xi \in X$.

5. [Hartshorne, Ex.II.1.3]

(a) Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Show that α is surjective iff the following condition holds: for every open $U \subset X$, and for every $t \in \mathcal{G}(U)$, there is an open covering $\{U_i\}_i$ of U, and sections $s_i \in \mathcal{F}(U_i)$, such that $\alpha(U_i)(s_i) = t|_{U_i}$ for all i.

(b) Give an example of a surjective morphism of sheaves $\alpha : \mathcal{F} \to \mathcal{G}$, and an open set U, such that $\alpha(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is not surjective.

6. [Hartshorne, Ex.II.1.19] *(Extending a sheaf by zero)* Let X be a topological space, let Z be a closed subset, let $U = X \setminus Z$, and let $i : Z \to X$ and $j : U \to X$ be the inclusions.

(a) Let $\mathcal{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathcal{F})_x$ of the direct image sheaf on X is $\mathcal{F}_x$ if $x \in Z$, and 0 if $x \notin Z$.

(b) Now let $\mathcal{F}$ be a sheaf on U. Let $j_!(\mathcal{F})$ be the sheaf on X obtained as a sheafification of the presheaf $\{V \mapsto \mathcal{F}(V)$ if $V \subset U$, $V \mapsto 0$ otherwise$\}$. Show that the stalk $(j_!(\mathcal{F}))_x$ is equal to $\mathcal{F}_x$ if $x \in U$, and 0 if $x \notin U$, and show that $j_!(\mathcal{F})$ is the only sheaf on X which has this property, and whose restriction to U is $\mathcal{F}$.

7. Prove Corollaries 8.28 and 8.33 from Lecture Notes.