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Complex Analytic Geometry

by

Janusz Adamus

Lecture Notes
PART I

Department of Mathematics
The University of Western Ontario

Contents

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1 Preliminaries

Some notation:

- $\mathbb{N} = \{0, 1, 2, \dots\}$; $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$.
- Given real $r > 0$ and $k \in \mathbb{N}_+$, $r\Delta^k = \{(z_1, \dots, z_k) \in \mathbb{C}^k \, : \, |z_i| < r\}$.
- $\mathbb{C}^0 = \{0\}$.

1.1 Complex manifolds

Definition 1.1. Given open subset $\Omega \subset \mathbb{C}^n$, a continuous function $f : \Omega \to \mathbb{C}$ is called *holomorphic in* Ω (or simply, *holomorphic*) when, for every $x = (x_1, \dots, x_n) \in \Omega$, there is a positive radius r and a sequence $(c_\nu)_{\nu \in \mathbb{N}^n} \subset \mathbb{C}$, such that

$$f(z) = \sum_{\nu \in \mathbb{N}^n} c_\nu (z - x)^\nu \quad \text{for all } z \in x + r\Delta^n \,,$$

where $(z - x)^\nu = (z_1 - x_1)^{\nu_1} \dots (z_n - x_n)^{\nu_n}$.
A mapping $f = (f_1, \dots, f_m) : \Omega \to \mathbb{C}^m$ is a *holomorphic mapping* when all its components are holomorphic functions.

Definition 1.2. Let $m \in \mathbb{N}$. A *complex manifold of dimension* m is a nonempty Hausdorff space M together with a complex *atlas* $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ satisfying:

(i) U_α nonempty and open in M for all $\alpha \in A$

(ii) $\varphi_\alpha : U_\alpha \to \mathbb{C}^m$ homeomorphism onto a region (open connected subset) in $\mathbb{C}^m$ for all $\alpha \in A$

(iii) $\bigcup_{\alpha \in A} U_\alpha = M$

(iv) $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is holomorphic for all $\alpha, \beta \in A$.

Two atlases on M are said to be *equivalent* when their union is an atlas itself. An equivalence class of all equivalent atlases is called a (*complex*) *structure* on M. A *coordinate chart* is any pair (U, φ) such that $\{(U, \varphi)\} \cup \mathcal{A}$ is an atlas. In particular, the $(U_\alpha, \varphi_\alpha)$ are coordinate charts.

Remark 1.3. It is sometimes convenient to treat the empty set as a manifold. It is, by definition, of dimension -1.

Definition 1.4. Given manifolds M and N and an open $\Omega \subset M$, a continuous mapping $f : \Omega \to N$ is called *holomorphic* when, for every pair of charts (U, φ) on M and (V, ψ) on N such that $f(U \cap \Omega) \subset V$, the composite

$$\psi \circ f \circ (\varphi|_{U \cap \Omega})^{-1} : \varphi(U \cap \Omega) \to \psi(V)$$

is holomorphic. We write $f \in \mathcal{O}(\Omega, N)$.

Remark 1.5.

(1) Given a nonempty manifold M and a nonempty open subset Ω in M, Ω is itself a manifold and $\dim \Omega = \dim M$. We call it an *open submanifold* of M. The structure is that restricted from M, that is, the equivalence class of the atlas $\{U_\alpha \cap \Omega, \varphi_\alpha|_\Omega\}_\alpha$ where $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is an atlas on M.

(2) For $n \in \mathbb{N}$, we regard $\mathbb{C}^n$ as a manifold equipped with the canonical complex structure induced by the identity mapping.

(3) When $N = \mathbb{C}$, we write $\mathcal{O}(\Omega)$ for $\mathcal{O}(\Omega, \mathbb{C})$. Pointwise addition and multiplication of functions define a commutative ring structure on $\mathcal{O}(\Omega)$. In fact, after identifying $\mathbb{C}$ with the constant functions, $\mathcal{O}(\Omega)$ is a $\mathbb{C}$-algebra. If Ω is conneted, then it follows from the Identity Principle below, that $\mathcal{O}(\Omega)$ is an integral domain: For $f \neq 0$ in $\mathcal{O}(\Omega)$, the set $f^{-1}(0)$ is nowhere dense in Ω, hence fg vanishes on Ω only if f or g does so.

Definition 1.6. A subset N of an m-dimensional manifold M is called a *submanifold* of dimension $n \leq m$ when, for every point $\xi \in N$, there is a coordinate chart (U, φ) on M such that $\xi \in U$ and

$$N \cap U = \varphi^{-1}(\{x = (x_1, \dots, x_m) \in \mathbb{C}^m \,:\, x_{n+1} = \dots = x_m = 0\})\,.$$

Of greatest interest to us will, in fact, be the submanifolds of $\mathbb{C}^n$. One easily proves the following useful characterisation of such submanifolds:

Proposition 1.7. *A subset $M \subset \mathbb{C}^n$ is an m-dimensional submanifold of $\mathbb{C}^n$ if and only if M is locally (at every point $\xi \in M$) a graph of a holomorphic function; that is, for every $\xi \in M$ there is an m-dimensional linear subspace $L \subset \mathbb{C}^n$, open neighbourhoods V of $\pi_L(\xi)$ in L and W of $\pi_{L^\perp}(\xi)$ in $L^\perp$, and a holomorphic $f : V \to W$, such that*

$$M \cap (V \times W) = \Gamma_f\,,$$

where Γ_f denotes the graph of f.

Theorem 1.8 (Identity Principle)**.** *Let M and N be complex manifolds, and let M be connected. Suppose $f, g \in \mathcal{O}(M, N)$, and there is a nonempty open subset $\Omega \subset M$ such that $f|_\Omega = g|_\Omega$. Then $f = g$.*

Proof. Put $D = \{\xi \in M \,:\, f|_W = g|_W$ for some open neighbourhood W of $\xi\}$. Then D is open and nonempty. We will show that D is also closed, which in light of connectedness of M will imply $D = M$. Let $\xi \in \overline{D}$. Then $f(\xi) = g(\xi)$, by continuity of f and g. Let (U, φ) and (V, ψ) be coordinate charts on M and N respectively, such that $\xi \in U$, $f(\xi) \in V$, and $f(U) \cup g(U) \subset V$. Then $\psi \circ f \circ \varphi^{-1}$ and $\psi \circ g \circ \varphi^{-1}$ agree on $\varphi(D \cap U)$. Since $D \cap U$ is a nonempty open subset of U, it follows from the Identity Principle for holomorphic functions in $\mathbb{C}^m$ that $\psi \circ f \circ \varphi^{-1}$ and $\psi \circ g \circ \varphi^{-1}$ agree on $\varphi(U)$. Thus $f|_U = g|_U$, and hence $\xi \in D$, as required. □

Definition 1.9. Let M and N be complex manifolds, of dimensions m and n respectively, and let $\xi \in M$. For a holomorphic mapping $f : M \to N$, the *rank of f at the point ξ*, denoted $\mathrm{rk}_\xi f$, is defined as the rank of the Jacobi matrix

$$\left[\frac{\partial F_i}{\partial x_j}(\varphi(\xi))\right]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}\,,$$

where $F = (F_1, \dots, F_n) = \psi \circ f|_U \circ \varphi^{-1} : \varphi(U) \to \psi(V)$, and (U, φ), (V, ψ) are coordinate charts around ξ and $f(\xi)$ respectively, such that $f(U) \subset V$.

The definition is independent of the choice of charts (**Exercise**). We say that the mapping $f : M \to N$ is of constant rank r when $\mathrm{rk}_x f = r$ for all $x \in M$. We have the following classical result (see, e.g., [Lo, C.4.1] for the proof):

Theorem 1.10 (Rank Theorem)**.** *Let $f : M \to N$ be a holomorphic map of constant rank r, and let $\xi \in M$. Then there exist coordinate charts (U, φ) and (V, ψ) in M and N respectively, such that $\xi \in U$, $f(U) \subset V$, and $\psi \circ f|_U \circ \varphi^{-1}$ is a linear map of rank r. Moreover, $f(U)$ is an r-dimensional submanifold of N, and nonempty fibres of $f|_U$ are $(m - r)$-dimensional submanifolds of M.*

1.2 Set and function germs

Definition 1.11. Let X be a topological space, and $\xi \in X$. Consider the following equivalence relation on $\mathcal{P}(X)$:

$$A \sim_\xi B \;\Leftrightarrow\; \text{there is an open nbhd } U \text{ of } \xi \text{ st. } A \cap U = B \cap U\,.$$

The elements of the quotient space $\mathcal{P}(X)/\sim_\xi$ are called the *set germs* at ξ. The equivalence class of A is denoted A_ξ, and A is a *representative* of A_ξ.

Remark 1.12.

(1) We say that $A_\xi \subset B_\xi$ when $A \subset B$ for some representatives A and B of A_ξ and B_ξ respectively.

(2) Finite set-theoretical operations commute with taking a germ at a point; e.g., $A_\xi \cup B_\xi = (A \cup B)_\xi$, $A_\xi \cap B_\xi = (A \cap B)_\xi$. We define the *product of germs* as $A_\xi \times B_\eta = (A \times B)_{(\xi,\eta)}$ (the definition is independent of the representatives chosen, **Exercise**).

(3) $A_\xi \neq \varnothing \;\Leftrightarrow\; \xi \in \overline{A}\,.$

(4) Representatives of X_ξ are precisely those sets $A \subset X$ that satisfy $\xi \in \mathrm{int}A$.

Definition 1.13. Let X be a topological space, $\xi \in X$, and let $\mathcal{F}(X,\xi)$ be the collection of all complex-valued functions (U,f) with domain U an open neighbourhood of ξ. Consider the following equivalence relation on $\mathcal{F}(X,\xi)$:

$$(U,f) \sim_\xi (V,g) \;\Leftrightarrow\; \text{there is an open nbhd } W \text{ of } \xi \text{ st. } f|_W = g|_W\,.$$

The elements of the quotient space $\mathcal{F}(X,\xi)/\sim_\xi$ are the *function germs* at ξ. The equivalence class of (U,f) is denoted f_ξ, and (U,f) is a *representative* of f_ξ.

Remark 1.14.

(1) The following are well-defined (i.e., independent of the choice of representatives, **Exercise**): $f_\xi \pm g_\xi = (f \pm g)_\xi$, $f_\xi \cdot g_\xi = (fg)_\xi$, and $f_\xi / g_\xi = (f/g)_\xi$ provided g is non-zero in a neighbourhood of ξ.

(2) We say that the function germ f_ξ *vanishes at a set germ* A_ξ, and write $f_\xi|_{A_\xi} = 0$, when some representative (U,f) of f_ξ vanishes on $A \cap U$, where A is a representative of A_ξ.

(3) **Warning!** The "image" $f_\xi(A_\xi)$ is in general not well-defined (**Exercise**).

1.3 Dimension

Definition 1.15 (Topological Dimension)**.** Let M be an m-dimensional complex[1] manifold, and let $A \subset M$. We define the (*topological*) *dimension* of A as $\dim \varnothing = -1$ and

$$\dim A = \max\{\dim N \,:\, N \text{ submanifold of } M, N \subset A\}$$

otherwise. For $\xi \in M$, the *dimension of A at the point ξ* is defined as

$$\dim_\xi A = \min\{\dim(A \cap U) \,:\, U \text{ open nbhd of } \xi\}\,.$$

[1] We will not mention the word *complex* again; all manifolds considered will be complex, unless otherwise specified.

One easily verifies (**Exercise**) the following:

Proposition 1.16. *Given a subset A of a manifold M, the function*

$$M \ni x \mapsto \dim_x A \in \mathbb{Z}$$

is upper semi-continuous; that is, for every $\xi \in M$ there is an open neighbourhood U such that $\dim_x A \leq \dim_\xi A$ for all $x \in U$.

Definition 1.17. Let M be a manifold, $\xi \in M$, and E a germ at ξ. We define the *dimension of the germ* E as $\dim E = \dim_\xi A$, where A is a representative of E at ξ.
(The above proposition guarantees that the definition is independent of the choice of representative.)

Remark 1.18.

(1) For a subset A of an m-dimensional manifold, we have $\dim A = m$ iff $\mathrm{int} A \neq \varnothing$. Similarly $\dim A_\xi = m$ iff $\mathrm{int} A \neq \varnothing$ for every representative A of A_ξ.

(2) In particular, $\dim A < m$ if A is nowhere dense.

(3) $A \subset B$ implies $\dim A \leq \dim B$. Similarly, $A_\xi \subset B_\xi$ implies $\dim A_\xi \leq \dim B_\xi$.

(4) If $A \subset M$ and $B \subset N$, then
$$\dim(A \times B) = \dim A + \dim B\,.$$
The inequality " $\geq$ " is clear, and the opposite one follows from the Rank Theorem 1.10 via the proposition below.

Proposition 1.19. *Let M, N be manifolds, let $\pi : M \times N \to M$ be the canonical projection, and let $E \subset M \times N$. If $\dim \pi^{-1}(z) \leq k$ for all $z \in \pi(E)$, then $\dim E \leq k + \dim \pi(E)$.*

Proof. Let Γ be a nonempty submanifold of E and let $\xi \in \Gamma$ be a point at which the rank of $\pi|_\Gamma : \Gamma \to M$ is maximal. Then $\pi|_\Gamma$ is of constant rank in a neighbourhood of ξ in Γ (by Example 2.5(5) and Theorem 2.7 below), so by the Rank Theorem, there is an open neighbourhood Γ_0 of ξ in Γ such that $\pi(\Gamma_0)$ is a submanifold of M, and nonempty fibres of $\pi|_{\Gamma_0}$ are submanifolds of dimension $r = \dim \Gamma_0 - \dim \pi(\Gamma_0)$, which is at most k, by assumption. Hence

$$\dim \Gamma = \dim \Gamma_0 = \dim \pi(\Gamma_0) + r \leq \dim \pi(\Gamma_0) + k \leq \dim \pi(E) + k\,,$$

and thus $\dim E = \max\{\dim \Gamma \,:\, \Gamma \subset E \text{ a submanifold of } M \times N\} \leq k + \dim \pi(E)\,.$ $\square$

2 Analytic sets

2.1 Analytic sets

Definition 2.1. Let Ω be an open subset of a complex manifold M. A set X is called an *analytic set* in Ω when, for every $\xi \in \Omega$, there is an open neighbourhood U of ξ in Ω and a finite collection of holomorphic functions $h_1, \dots, h_s \in \mathcal{O}(U)$ such that

$$X \cap U = \{x \in U : h_1(x) = \cdots = h_s(x) = 0\} \;=\; h_1^{-1}(0) \cap \cdots \cap h_s^{-1}(0)\,.$$

X is called *locally analytic* in Ω when there are a neighbourhood U and functions $h_1, \dots, h_s$ as above for every $\xi \in X$ (but not necessarily for $\xi \in \Omega \setminus X$).

Remark 2.2.

(1) Since an open subset of a manifold is a manifold itself, we can simply speak of [locally] analytic subsets of manifolds.

(2) X is analytic in Ω iff X is locally analytic and closed in Ω.

(3) Every locally analytic set X in Ω is analytic in some open $U \subset \Omega$ (for instance, in $\Omega \setminus (\overline{X} \setminus X)$). $\overline{X} \setminus X$ is called the *frontier* of X.

Example 2.3. $X = \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 < 1\}$ is locally analytic and not analytic in $\mathbb{C}^2$: for every $\xi \in X$, $X \cap U = h^{-1}(0)$, for U an open disc centered at ξ and contained in X, and $h \equiv 0$.

Remark 2.4. One also defines *globally analytic* sets (but these will be of no interest to us): X is *globally analytic* in M when there is a finite collection of holomorphic functions $h_1, \dots, h_s \in \mathcal{O}(M)$ such that $X = \{x \in M : h_1(x) = \cdots = h_s(x) = 0\}$.

Example 2.5.

1. The union of a collection of some connected components of a manifold M is analytic in M.
2. A closed submanifold N of M is analytic in M.
3. An algebraic subset of $\mathbb{C}^n$ (i.e., the locus of common zeros of a collection of polynomials in n complex variables) is globally analytic in $\mathbb{C}^n$.
4. Nonempty proper analytic subsets of $\mathbb{C}$ (and, in general, of one-dimensional manifolds) are precisely the sets of isolated points without an accumulation point. Locally analytic subsets are the sets of isolated points (e.g., $\{1/n : n \in \mathbb{N}_+\}$).
5. Let $f : M \to N$ be a holomorphic mapping of manifolds, and let $k \in \mathbb{N}$. Then the set

$$\{x \in M : \mathrm{rk}_x f \le k\}$$

is analytic in M. **Exercise**.

2.2 Basic topological properties

Union

If $\{X_\lambda\}_{\lambda\in\Lambda}$ is a locally finite family of analytic subsets of a manifold M, then $\bigcup_{\lambda\in\Lambda} X_\lambda$ is analytic in M. Notice that even finite unions of locally analytic sets need not be locally analytic.

Example 2.6. Consider the union of the set X of Example 2.3 and the singleton $Y = \{(1,0)\}$ (Y is in fact globally analytic) in $\mathbb{C}^2$. Then $X \cup Y$ does not satisfy the definition of local analycity at the point $(1,0)$ (by the Identity Principle).

Intersection

A finite intersection of [locally] analytic subsets of a manifold M is [locally] analytic in M itself. In fact, *arbitrary* intersections of [locally] analytic sets are [locally] analytic, as we will show later (Theorem 7.11).

Cartesian product

If X_1 (resp. X_2) is [locally] analytic in a manifold M_1 (resp. M_2), then $X_1 \times X_2$ is [locally] analytic in $M_1 \times M_2$.

Inverse image

If $\varphi : M \to N$ is a holomorphic mapping of manifolds, and Y is [locally] analytic in N, then $\varphi^{-1}(Y)$ is [locally] analytic in M.

Complement

Theorem 2.7. *Let M be a connected manifold, and let X be a proper analytic subset of M. Then $M \setminus X$ is an open, arcwise connected, dense subset of M.*

Proof. Openness is clear. For the proof of density, suppose that $M \setminus X$ is not dense, so that $D := \operatorname{int} X$ is not empty. We will show that $D = \overline{D}$, and hence $D = M$, a contradiction. Let $\xi \in \partial D$. Then $\xi \in X$, as X is closed, and so there exists an open neighbourhood U of ξ and functions $h_1, \dots, h_s \in \mathcal{O}(U)$ such that $X \cap U = \{x \in U : h_1(x) = \dots = h_s(x) = 0\} \neq \varnothing$. Then $h_1, \dots, h_s$ all vanish on $D \cap U$, which by the Identity Principle implies $h_i \equiv 0$ $(i = 1, \dots, s)$, $D \cap U$ being nonempty and open in U. Therefore $X \cap U = U$, and hence $\xi \in D$, as required.

To prove that $M \setminus X$ is arcwise connected, suppose first that (U, φ) is a coordinate chart in M. Without loss of generality, we may assume that U is an open ball in $\mathbb{C}^n$. Let a, b be two points in $U \setminus X$, and let L be a complex line through a and b. Then, by Example 2.5(4), $X \cap U \cap L$ is a set without accumulation points in $L \cap U$, and hence a and b can be joined by an arc in $U \setminus X$. In the general case, M being arcwise connected, for any two points a and b in $M \setminus X$, we can find an arc $\gamma : [0,1] \to M$ with $\gamma(0) = a$ and $\gamma(1) = b$. Then γ has a finite cover by coordinate charts (U_j, φ_j) $(j = 1, \dots, s)$ such that $a \in U_1$, $b \in U_s$, and every $U_j \setminus X$ is arcwise connected, open and dense. We can thus patch an $a - b$ arc in $M \setminus X$ from finitely many pieces. $\square$

2.3 Regular and singular points

Definition 2.8. Let X be an analytic subset of a manifold M. A point $\xi \in X$ is called *regular* (or *smooth*), and X is called *regular* (or *smooth*, or *non-singular*) *at* ξ, when ξ has an open neighbourhood U in M such that $X \cap U$ is a submanifold of M. The set of regular points of X is denoted by $\mathrm{reg}X$ (or X^{reg}, or X^0 [Ło], or X^- [Wh]).

The set $X \setminus \mathrm{reg}X$ is denoted by $\mathrm{sng}X$ (or X^{sng}, or X^* [Ło], or $X^\times$ [Wh]), and called the *singular locus* (or the *set of singular points*) of X.

Example 2.9.

1. For $X = \{(x,y) \in \mathbb{C}^2 : xy = 0\}$, we have $\mathrm{sng}X = \{0\}$ (not a graph of a function at 0 - fails the vertical line test).
2. For $X = \{(x,y) \in \mathbb{C}^2 : x^2 = y^3\}$, also $\mathrm{sng}X = \{0\}$ (unlike in the real case, not a function at 0).
3. For $X = \{(x,y,x) \in \mathbb{C}^3 : xy = 0\}$, $\mathrm{sng}X = \{0\} \times \mathbb{C}$.
4. If $f : M \to N$ is a biholomorphism, X analytic in M, then ξ is a regular (resp. singular) point of X iff $f(\xi)$ is regular (resp. singular) for $f(X)$.

Proposition 2.10. *If X is an analytic subset of a one-dimensional manifold M, then* $\mathrm{reg}X = X$.

Proof. Let $\xi \in X$ and let U be a connected neighbourhood of ξ on which there live $h_1, \dots, h_s \in \mathcal{O}(U)$ such that $X \cap U = h_1^{-1}(0) \cap \dots \cap h_s^{-1}(0)$. If $X \cap U = U$, then $\xi \in \mathrm{reg}X$. If, in turn, $X \cap U \subsetneq U$, then by Example 2.5(4), $X \cap U$ is a set of isolated points. Thus, after shrinking U, if needed, $X \cap U = \{\xi\}$ is a 0-dimensional manifold in M, so $\xi \in \mathrm{reg}X$ again. □

Theorem 2.11. *Let X be an analytic subset of a manifold M. Then* $\mathrm{reg}X$ *is open and dense in X.*

Proof. The openness is clear. We prove density by induction on $m = \dim M$. The case $m = 1$ is done above, so let's assume the statement holds for $m - 1$, and consider an analytic set X in an m-dimensional manifold M ($m > 1$). Let $\xi \in X$. We will show that an arbitrarily small neighbourhood U of ξ in M intersects $\mathrm{reg}X$. Let U be an open neighbourhood of ξ, small enough so that there exist $h_1, \dots, h_s \in \mathcal{O}(U)$ for which $X \cap U = h_1^{-1}(0) \cap \dots \cap h_s^{-1}(0)$. Without loss of generality, we may assume that U is a domain of a coordinate chart, and hence that U is a connected open subset of $\mathbb{C}^m$.

Now, if $X \cap U = U$, then $\xi \in \mathrm{reg}X$, so we may assume that $X \cap U \subsetneq U$. Then at least one of the functions $h_1, \dots, h_s$ doesn't vanish identically on U; say, $h_1 \not\equiv 0$. By the Identity Principle, there exists a multiindex $\alpha \in \mathbb{N}^m$ such that $\left(\frac{\partial^{|\alpha|} h_1}{\partial x^\alpha}\right)(\xi) \neq 0$. Hence there exist a point $\eta \in X \cap U$, a multiindex $\beta \in \mathbb{N}^m$ and $1 \leq j \leq m$ for which

$$\left.\frac{\partial^{|\beta|} h_1}{\partial x^\beta}\right|_{X \cap U} \equiv 0 \quad \text{and} \quad \frac{\partial}{\partial x_j}\left(\frac{\partial^{|\beta|} h_1}{\partial x^\beta}\right)(\eta) \neq 0\,.$$

Then, in some coordinate neighbourhood $V \subset U$ of η, the function $\left.\frac{\partial}{\partial x_j}\left(\frac{\partial^{|\beta|} h_1}{\partial x^\beta}\right)\right|_V$ is never 0. Put $g = \partial^{|\beta|} h_1 / \partial x^\beta$. Then $g \in \mathcal{O}(V)$, and as $\partial g / \partial x_j$ is never zero on V, $g^{-1}(0)$ is an $(m-1)$-dimensional submanifold of V. But $X \cap V \subset g^{-1}(0)$, as $g|_{X \cap V} \equiv 0$, and so by the inductive hypothesis, $V \cap g^{-1}(0)$ (and hence U itself) contains regular points of X. □

2.4 Principal analytic sets

Definition 2.12. An analytic subset X of a manifold M is called [*locally*] *principal* when it is [locally] the zero set of a single function h, which doesn't vanish identically on any connected component of M.

Theorem 2.13. *Let X be a locally principal analytic subset of an m-dimensional manifold M. Then X is of pure dimension $m-1$.*

Proof. Since $\mathrm{int}X = \varnothing$, by Theorem 2.7, it follows that $\dim_x X \leq m-1$ at every $x \in X$. To prove the opposite inequality, it suffices to show that $\mathrm{reg}X$ is an $(m-1)$-dimensional submanifold of M. Consider N a connected component of $\mathrm{reg}X$. We have $\dim N = k \leq m-1$. Suppose $k < m-1$, and let $\xi \in N$. Then there exists a complementary submanifold T in (a neighbourhood of ξ in) M, of dimension $m-k \geq 2$, such that $N \cap T = \{\xi\}$. Let $h \in \mathcal{O}(U)$ be such that $X \cap U = h^{-1}(0)$ for some small neighbourhood U of ξ. Then $1/h \in \mathcal{O}(U \cap T \setminus \{\xi\})$ and the singleton $\{\xi\}$ is of codimension at least 2 in $U \cap T$. Therefore, by Hartogs' Lemma (see [Ło, C.1.11]), $1/h \in \mathcal{O}(U \cap T)$, and hence $h(\xi) \neq 0$; a contradiction. □

2.5 Irreducible analytic sets

Definition 2.14. An analytic subset X of a manifold M is called *reducible* when there exist nonempty analytic proper subsets X_1, X_2 of X such that $X = X_1 \cup X_2$. Otherwise X is called *irreducible.*

Proposition 2.15. *Let $f : M \to N$ be a holomorphic mapping of manifolds, let X be analytic in M and such that $Y = \varphi(X)$ is analytic in N. Then, if X is irreducible, then so is Y.*

Proof. Suppose $Y = Y_1 \cup Y_2$ for some nonempty analytic $Y_j \neq Y$ $(j = 1, 2)$. Then $f^{-1}(Y_j)$ are analytic in M, $X = X \cap f^{-1}(Y) = (X \cap f^{-1}(Y_1)) \cup (X \cap f^{-1}(Y_2))$, and $X = X \cap f^{-1}(Y) \neq X \cap f^{-1}(Y_j)$ $(j = 1, 2)$; a contradiction. □

Example 2.16.

1. $X = \{xy = 0\}$ is reducible.
2. $X = \{x^2 = y^3\}$ is irreducible, by Proposition 2.15, as the image of (irreducible) $\mathbb{C}$ under the parametrisation $\mathbb{C} \ni t \mapsto (t^3, t^2) \in \mathbb{C}^2$.

Proposition 2.17. *Let N be a closed submanifold of a manifold M. Then N is an irreducible analytic subset of M iff N is connected.*

Proof. Suppose that N is connected and X_1 and X_2 are nonempty analytic subsets of M, such that $X_1 \cup X_2 = N$. By Theorem 2.7, for $j = 1, 2$, either $X_j \cap N = N$ or else it is a nowhere-dense subset of N. By the Baire's Category Theorem, N is not a union of two nowhere-dense subsets, and hence one of the $X_j \cap N$ is not a proper subset of N.

If, in turn, a closed submanifold N is not connected, let X_1 be a component of N and let X_2 be the union of its remaining components. Then X_1 and X_2 are analytic in M (by Example 2.5(1),(2)), nonempty proper subsets of N, and $N = X_1 \cup X_2$. □

3 Hironaka division

3.1 Hironaka division theorem

We will be interested in division in $\mathbb{C}[[x]]$ and $\mathbb{C}\{x\}$, where $x = (x_1, \dots, x_m)$ is a system of m complex variables. Recall that $\mathbb{C}[[x]]$ is the ring of formal power series, and $\mathbb{C}\{x\}$ its subring of convergent power series in x. We will, in fact, consider more general coefficient rings than $\mathbb{C}$, namely $A = \mathbb{C}[[y]]/I$ or $\mathbb{C}\{y\}/I$, where I is a proper ideal, and $y = (y_1, \dots, y_n)$. Rings of the latter type are called *local analytic* $\mathbb{C}$*-algebras*. By definition, if $A = \mathbb{C}[[y]]/I$ (resp. $\mathbb{C}\{y\}/I$), then

$$A[[x]] = \frac{\mathbb{C}[[y,x]]}{I\cdot\mathbb{C}[[y,x]]} \qquad (\text{ resp. } \ A\{x\} = \frac{\mathbb{C}\{y,x\}}{I\cdot\mathbb{C}\{y,x\}}\,).$$

Let $H \in A[[x]]$. We write $H = \sum_{\beta\in\mathbb{N}^m} h_\beta x^\beta$, where $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{N}^m$, $x^\beta = x_1^{\beta_1} \dots x_m^{\beta_m}$, and $h_\beta \in A$.
Lexicographic ordering of the $(m+1)$-tuples

$$(|\beta|, \beta_1, \dots, \beta_m)\,,$$

where $|\beta| = \beta_1 + \dots + \beta_m$, defines a total ordering of $\mathbb{N}^m$ (or, equivalently, of all the monic monomials $x^\beta \in \mathbb{C}[[x]]$). This can be extended to a total ordering μ of the monomials $h\cdot x^\alpha \in A[[x]]$ (see below).

The mapping $\mathbb{C}[[y]] \to \mathbb{C}$ of evaluation at 0 induces an evaluation mapping

$$A \to \mathbb{C}\,, \qquad h(y) \mapsto h(0)\,,$$

and hence

$$A[[x]] \to \mathbb{C}[[x]]\,, \qquad H = \sum_{\beta\in\mathbb{N}^m} h_\beta x^\beta \ \ \mapsto \ \ H(0) = \sum_{\beta\in\mathbb{N}^m} h_\beta(0) x^\beta\,.$$

Given $H = \sum h_\beta x^\beta$, define the *support* of H

$$\mathrm{supp}(H) = \{\beta \in \mathbb{N}^m : h_\beta \not\equiv 0\}\,,$$

the *initial exponent* of H

$$\exp_\mu(H) = \min_\mu\{\beta : \beta \in \mathrm{supp}(H)\}\,,$$

where the minimum is taken with respect to the total ordering μ, and the *initial form* of H as

$$\mathrm{in}_\mu(H) = h_{\beta^0} x^{\beta^0}\,, \qquad \text{where} \ \ \beta^0 = \exp_\mu(H)\,.$$

Similarly, define $\mathrm{supp}(H(0)) = \{\beta \in \mathbb{N}^m : h_\beta(0) \neq 0\}$, $\exp(H(0))$, and $\mathrm{in}(H(0))$.

Given $G_i = \sum_\beta g^i_\beta x^\beta \in A[[x]]$, $i = 1, \dots, t$, consider the following partition of $\mathbb{N}^m$:
Put $\beta^i = \exp(G_i(0))$, $i = 1, \dots, t$, and let

$$\Delta_1 = \beta^1 + \mathbb{N}^m \qquad \text{and} \qquad \Delta_i = (\beta^i + \mathbb{N}^m) \setminus \bigcup_{1\le j<i} \Delta_j \quad \text{for } 1 < i \le t\,.$$

Finally, let

$$\Delta = \mathbb{N}^m \setminus \bigcup_{1\le j\le t} \Delta_j\,.$$

Next, we want to define a total ordering of all the "monomials" $h\cdot x^\beta \in A[[x]]$, suitable for our choice of $G_1, \dots, G_t$ above. Let $\mathfrak{m} = \mathfrak{m}_n$ denote the maximal ideal of $\mathbb{C}[[y]] = \mathbb{C}[[y_1, \dots, y_n]]$. If $h \in A$, the *order* of h, denoted $\nu(h)$, is defined as the largest $\nu \in \mathbb{N}$ such that $h \in (\mathfrak{m}^\nu + I)/I$; $\nu(h) = +\infty$ if $h \in I$. Consider the coefficients $g^i_{\beta^i}$ of the G_i, $i = 1, \dots, t$, where $\beta^i = \exp(G_i(0))$. Since $g^i_{\beta^i}(0) \neq 0$, they are all invertible elements of A, and hence without loss of generality, we may assume that $g^i_{\beta^i} \equiv 1$, $i = 1, \dots, t$. For $l \in \mathbb{N}$, we define a μ_l*-ordering* of monomials in $A[[x]]$, by lexicographic ordering the $(m + 1)$-tuples

$$\mu_l(h\cdot x^\beta) = (l\cdot\nu(h) + |\beta|,\, \beta_1,\, \dots,\, \beta_m)\,.$$

Lemma 3.1. *There exists* $l \in \mathbb{N}$ *such that* $\mu_l(x^{\beta^i}) < \mu_l(g^i_\beta x^\beta)$ *whenever* $\beta < \beta^i$, $i = 1, \dots, t$. *(For such* l*, we have* $\exp_{\mu_l}(G_i) = \exp(G_i(0))$, $i = 1, \dots, t$*).*

Proof. We want to have $|\beta^i| < l\cdot\nu(g^i_\beta) + |\beta|$ whenever $\beta < \beta^i$. Notice that, for such β, $\nu(g^i_\beta) > 0$ (by the choice of β^i). Therefore, if $|\beta| = |\beta^i|$, any positive l will do. If, in turn, $|\beta| < |\beta^i|$, we need $l > (|\beta^i| - |\beta|)/\nu(g^i_\beta)$. Since there is only finitely many such β's, we may choose $l \in \mathbb{N}$ such that

$$\frac{1}{l} < \min_{i=1,\dots,t}\left(\min_{|\beta|<|\beta^i|} \frac{\nu(g^i_\beta)}{|\beta^i| - |\beta|}\right)\,.$$

□

Example 3.2. Let $G(y, x_1, x_2) = yx_1 + x_1x_2^2 + e^y x_1^3$. Then $G(0) = x_1x_2^2 + x_1^3$, hence $\exp(G(0)) = (1, 2)$ (the lexicographic minimum of $(1, 2)$ and $(3, 0)$). The exponent at x of the monomial yx_1 is $(1, 0)$ and the order of its coefficient $h(y) = y$ is 1. The other (infinitely many) monomials of G are negligable, since each of them has the exponent at x equal to $(3, 0)$, which is strictly greater than $(2, 1)$. Therefore we need to choose l so that

$$3 = |(2, 1)| < l \cdot 1 + |(1, 0)| = l + 1\,.$$

Thus $l = 3$ will do.

Theorem 3.3 (Hironaka Division)**.** *Let* $F, G_1, \dots, G_t \in A[[x]]$*, let* $\beta^i = \exp(G_i(0))$ *and* Δ, Δ_i *be as above. Then there exist unique* $Q_1, \dots, Q_t, R \in A[[x]]$ *such that*

$$F = \sum_{i=1}^t Q_i G_i + R,$$

where $\beta^i + \mathrm{supp}(Q_i) \subset \Delta_i$ *and* $\mathrm{supp}(R) \subset \Delta$.

Proof. First assume that $A = \mathbb{C}$. For the proof of uniqueness, suppose $\sum Q_i G_i + R = \sum Q'_i G_i + R'$. Then $\sum(Q_i - Q'_i)G_i = R' - R$, and the initial exponents of the left and right hand side (if not zero) belong to disjoint regions of $\mathbb{N}^m$, which is impossible. Hence $Q_i = Q'_i$ $(i = 1, \dots, t)$ and $R' = R$.

Now for the division algorithm: By collecting all terms of F divisible by x^{β^1} (and factoring out of them x^{β^1}), then all of the remaining terms divisible by x^{β^2} (and factoring out of them x^{β^2}), and so on until no such term remains, we get respectively $Q_1(F), \dots, Q_t(F)$, and $R(F) \in \mathbb{C}[[x]]$ such that

$$F = \sum_{i=1}^t Q_i(F)x^{\beta^i} + R(F), \text{ where } \beta^i + \mathrm{supp}(Q_i(F)) \subset \Delta_i \text{ and } \mathrm{supp}(R(F)) \subset \Delta\,.$$

Hence also $\exp(Q_i(F)) \geq \exp(F) - \beta^i$, and $\exp(R(F)) \geq \exp(F)$.
Put $E(F) = F - \sum Q_i(F)G_i - R(F)$; i.e.,

$$E(F) = \sum Q_i(F)(x^{\beta^i} - G_i)\,.$$

Then

$$\exp(E(F)) = \min_i\{\exp(Q_i(F)\cdot(x^{\beta^i} - G_i))\} > \min_i\{\exp(Q_i(F)\cdot x^{\beta^i})\} \geq \exp(F)\,,$$

where the equality holds because these summands of $E(F)$ are supported in disjoint regions of $\mathbb{N}^m$, and the strict inequality follows from the fact that $\mathrm{in}(G_i) = x^{\beta^i}$.

Now, as for F before, there exist $Q_i(E(F))$, $i = 1, \dots, t$, and $R(E(F))$ such that

$$E(F) = \sum_{i=1}^{t} Q_i(E(F))x^{\beta^i} + R(E(F)),\ \text{ where } \beta^i + \mathrm{supp}(Q_i(E(F))) \subset \Delta_i \text{ and } \mathrm{supp}(R(E(F))) \subset \Delta\,.$$

As before, we get $\exp(Q_i(E(F))) \geq \exp(E(F)) - \beta^i$, and $\exp(R(E(F))) \geq \exp(E(F))$.
Put $E^2(F) = E(F) - \sum Q_i(E(F))G_i - R(E(F))$; i.e.,

$$E^2(F) = \sum Q_i(E(F))(x^{\beta^i} - G_i)\,.$$

Then

$$\exp(E^2(F)) = \min_i\{\exp(Q_i(E(F))\cdot(x^{\beta^i} - G_i))\} > \min_i\{\exp(Q_i(E(F))\cdot x^{\beta^i})\} \geq \exp(E(F))\,.$$

And so on... Having defined $Q_i(E^k(F))$ and $R(E^k(F))$ for all $k \in \mathbb{N}$, let

$$Q_i = \sum_{k=0}^{\infty} Q_i(E^k(F)) \quad \text{and} \quad R = \sum_{k=0}^{\infty} R(E^k(F))\,, \tag{3.1}$$

where $E^0(F) = F$ and $E^{k+1}(F) = E(E^k(F))$ as above.

Now, $\beta^i + \mathrm{supp}(Q_i) \subset \Delta_i$ and $\mathrm{supp}(R) \subset \Delta$, because $\beta^i + \mathrm{supp}(Q_i(E^k(F))) \subset \Delta_i$ and $\mathrm{supp}(R(E^k(F))) \subset \Delta$ for all $k \in \mathbb{N}$. Moreover, the two series in (3.1) converge in Krull topology of $\mathbb{C}[[x]]$, as

$$\exp(Q_i(E^k(F))) \geq \exp(E^k(F)) - \beta^i > \exp(E^{k-1}(F)) - \beta^i$$
$$\text{and} \quad \exp(R(E^k(F))) \geq \exp(E^k(F)) > \exp(E^{k-1}(F))\,.$$

Finally,

$$F - \sum_{i=1}^{t}\left(\sum_{k=0}^{l} Q_i(E^k(F))\right)G_i - \left(\sum_{k=0}^{l} R(E^k(F))\right) =$$
$$\left[F - \sum_{i=1}^{t} Q_i(F)G_i - R(F)\right] - \left[\sum_{i=1}^{t} Q_i(E(F))G_i + R(E(F))\right] - \dots - \left[\sum_{i=1}^{t} Q_i(E^l(F))G_i + R(E^l(F))\right] = E^{l+1}(F)\,,$$

and hence, by convergence in (3.1),

$$F - \sum_{i=1}^{t} Q_iG_i - R = \lim_{l\to\infty}\left[F - \sum_{i=1}^{t}\left(\sum_{k=0}^{l} Q_i(E^k(F))\right)G_i - \left(\sum_{k=0}^{l} R(E^k(F))\right)\right] = \lim_{l\to\infty} E^{l+1}(F) = 0\,,$$

since $\lim_{l\to\infty} \exp(E^{l+1}(F)) = \infty$. Therefore $F = \sum_i Q_i G_i + R$.

Now, for general $A = \mathbb{C}[[y]]/I$, taking l as in Lemma 3.1 above, we complete the proof of Theorem 3.3 with the μ_l-ordering: The initial form of G_i with respect to this ordering is still x^{β^i}, since $\nu(1) = 0$. The algorithm is the same as before; the uniqueness and (Krull) convergence arguments are also the same, using the new ordering: $\exp_{\mu_l}(Q_i(F)x^{\beta^i}) \geq \exp_{\mu_l}(F)$, $\exp_{\mu_l}(R(F)) \geq \exp_{\mu_l}(F)$, and $\exp_{\mu_l}(x^{\beta^i} - G_i) > \exp_{\mu_l}(G_i)$, by the choice of l, so

$$\exp_{\mu_l}(E(F)) \geq \min_i\{\exp_{\mu_l}(Q_i(F)\cdot(x^{\beta^i} - G_i)) > \exp_{\mu_l}(F)\,.$$

□

For the convergent version of Hironaka's division theorem, we need the following $\rho{-}\sigma$-norms: If $h = h(y) = \sum_{\alpha\in\mathbb{N}^n} h_\alpha y^\alpha \in A = \mathbb{C}\{y\}/I$, put

$$\|h\|_\rho = \sum_{\alpha\in\mathbb{N}^n} |h_\alpha|\cdot\rho^{|\alpha|}\,, \qquad \text{where } \rho > 0\,,$$

and, for $H = \sum_{\beta\in\mathbb{N}^m} h_\beta x^\beta \in A\{x\}$, put

$$\|H\|_{\rho,\sigma} = \sum_{\beta\in\mathbb{N}^m} \|h_\beta\|_\rho\cdot\sigma^{|\beta|}\,, \qquad \text{where } \sigma > 0\,.$$

Remark 3.4. It follows directly from the definition that, for $\rho, \sigma > 0$ and arbitrary $H_1, H_2 \in A[[x]]$,

(1) $\|H_1H_2\|_{\rho,\sigma} \leq \|H_1\|_{\rho,\sigma}\cdot\|H_2\|_{\rho,\sigma}$.

(2) $\|H_1 + H_2\|_{\rho,\sigma} \leq \|H_1\|_{\rho,\sigma} + \|H_2\|_{\rho,\sigma}$, with equality if $\operatorname{supp}(H_1) \cap \operatorname{supp}(H_2) = \varnothing$.

Moreover, it is not difficult to verify the following

Proposition 3.5. *Let $H = \sum h_\beta x^\beta \in A[[x]]$. Then $H \in A\{x\}$ iff there exist positive ρ and σ such that $\|H\|_{\rho,\sigma} < \infty$.*

Proof. Indeed, if $H = \sum_{\beta\in\mathbb{N}^m} h_\beta x^\beta \in A[[x]]$, where $h_\beta = \sum_{\alpha\in\mathbb{N}^n} h^\beta_\alpha y^\alpha$, then $H \in A\{x\}$ iff there is a positive M such that $|h^\beta_\alpha| \in O(M^{|\alpha|+|\beta|})$ for all $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^m$. The proposition follows easily (**Exercise**). □

Theorem 3.6 (Convergent Hironaka Division)**.** *Under the notation of the previous theorem, if $F, G_1, \ldots, G_t \in A\{x\}$, then the unique $Q_1, \ldots, Q_t, R$ are also convergent.*

Proof. We will first show that there exists a $\sigma > 0$ such that there is a $\rho > 0$ for which

$$\left\|x^{\beta^i} - G_i\right\|_{\rho,\sigma} \leq \frac{1}{2}\,\sigma^{|\beta^i|} \qquad \text{for } \ i = 1, \ldots, t\,. \tag{3.2}$$

For this, consider the series $\sum_{|\beta|>|\beta^i|} \left\| g^i_\beta \right\|_{\rho'} \cdot \sigma^{|\beta|-|\beta^i|}$. We can choose $\rho' > 0$ so that it is a convergent power series in σ. As the series vanishes for $\sigma = 0$, one can choose $\sigma > 0$ small enough so that

$$\sum_{|\beta|>|\beta^i|} \left\| g^i_\beta \right\|_{\rho'} \cdot \sigma^{|\beta|-|\beta^i|} \;\le\; \frac{1}{4}\,. \tag{3.3}$$

For this choice of σ, we claim there exists $0 < \rho < \rho'$ such that

$$\sum_{|\beta|<|\beta^i|} \left\| g^i_\beta \right\|_{\rho} \cdot \sigma^{|\beta|-|\beta^i|} \;\le\; \frac{1}{4}\,. \tag{3.4}$$

Indeed, the sum is finite, and since $g^i_\beta(0) = 0$ for all β satisfying $|\beta| < |\beta^i|$, we have $\lim_{\rho\to 0} \left\| g^i_\beta \right\|_\rho = 0$. Combining (3.3) and (3.4) we get (3.2).

By Proposition 3.5, we may now choose $\sigma, \rho > 0$, such that $\|F\|_{\rho,\sigma}$, and $\|G_i\|_{\rho,\sigma}$, $i = 1, \dots, t$, are all finite, and (3.2) holds. Divide F as in the proof of Theorem 3.3: $F = \sum_{i=1}^{t} Q_i(F) x^{\beta^i} + R(F)$. By Remark 3.4(2),

$$\|F\|_{\rho,\sigma} = \sum_{i=1}^{t} \|Q_i(F)\|_{\rho,\sigma} \cdot \sigma^{|\beta^i|} + \|R(F)\|_{\rho,\sigma}\ .$$

Hence, by (3.2),

$$\|E(F)\|_{\rho,\sigma} \le \sum_{i=1}^{t} \|Q_i(F)\|_{\rho,\sigma} \cdot \frac{1}{2}\,\sigma^{|\beta^i|} \le \frac{1}{2}\,\|F\|_{\rho,\sigma}\ .$$

One shows recursively that

$$\left\|Q_i(E^k(F))\right\|_{\rho,\sigma} \le \sigma^{-|\beta^i|} \cdot \left\|E^k(F)\right\|_{\rho,\sigma} \le \frac{1}{2^k}\,\sigma^{-|\beta^i|} \cdot \|F\|_{\rho,\sigma}\ ,$$

$$\text{and}\qquad \left\|R(E^k(F))\right\|_{\rho,\sigma} \le \left\|E^k(F)\right\|_{\rho,\sigma} \le \frac{1}{2^k}\,\|F\|_{\rho,\sigma}\ .$$

Therefore

$$\|Q_i\|_{\rho,\sigma} \le \sum_{k=0}^{\infty} \frac{1}{2^k}\,\sigma^{-|\beta^i|} \cdot \|F\|_{\rho,\sigma} = 2\sigma^{-|\beta^i|} \cdot \|F\|_{\rho,\sigma}\ ,$$

and

$$\|R\|_{\rho,\sigma} \le \sum_{k=0}^{\infty} \frac{1}{2^k}\,\|F\|_{\rho,\sigma} = 2\,\|F\|_{\rho,\sigma}\ .$$

□

3.2 Weierstrass preparation theorem

Definition 3.7. Let $f(z,w) \in \mathbb{C}\{z,w\} = \mathbb{C}\{z_1,\dots,z_m,w\}$. We say that f is *regular of order d in w* when $f(0,w) = \varphi(w)\cdot w^d$ with $\varphi(0) \neq 0$. We say that f is *regular in w*, if it is regular of order d in w for some d.

Theorem 3.8 (Weierstrass Division Theorem)**.** *Suppose $f(z,w) \in \mathbb{C}\{z,w\}$ (resp. $\mathbb{C}[[z,w]]$) is regular of order d in w, and let $g(z,w) \in \mathbb{C}\{z,w\}$ (resp. $\mathbb{C}[[z,w]]$). Then there exist unique $q(z,w) \in \mathbb{C}\{z,w\}$ (resp. $\mathbb{C}[[z,w]]$) and $r_j(z) \in \mathbb{C}\{z\}$ (resp. $\mathbb{C}[[z]]$) ($j = 1,\dots,d$) such that*

$$g(z,w) = q(z,w)\cdot f(z,w) + \sum_{j=1}^{d} r_j(z)\cdot w^{d-j}\,.$$

Proof. We apply Theorem 3.6 (resp. 3.3) with $A = \mathbb{C}\{z\}$, $m = 1$, $F = g$, $t = 1$, and $G_1 = f$. Since $\mathrm{in}(f(0)) = \varphi(0)w^d$, then $\Delta_1 = d + \mathbb{N}$, $\Delta = \{0,1,\dots,d-1\}$, and

$$g(z,w) = q(z,w)\cdot f(z,w) + \sum_{j=1}^{d} r_j(z)\cdot w^{d-j}\,.$$

□

Let $P(z,w) = w^d + \sum_{j=1}^{d} a_j(z)\cdot w^{d-j} \in \mathbb{C}\{z\}[w]$ (resp. $\mathbb{C}[[z]][w]$). We say that P is a *distinguished polynomial* of degree d when $a_j(0) = 0$, $j = 1,\dots,d$.

Theorem 3.9 (Weierstrass Preparation Theorem)**.** *If $f(z,w) \in \mathbb{C}\{z,w\}$ (resp. $\mathbb{C}[[z,w]]$) is regular of order d in w, then there exist a distinguished polynomial $P(z,w) \in \mathbb{C}\{z\}[w]$ (resp. $\mathbb{C}[[z]][w]$) of degree d, and $q(z,w) \in \mathbb{C}\{z,w\}$ (resp. $\mathbb{C}[[z,w]]$) such that*

$$q(0,0) \neq 0 \qquad \text{and} \qquad f = q\cdot P\,.$$

Moreover, P and q are uniquely determined by these conditions.

Proof. By Theorem 3.8,

$$w^d = h(z,w)\cdot f(z,w) + \sum_{j=1}^{d} r_j(z)\cdot w^{d-j}\,.$$

Put $z = 0$ to get

$$w^d = h(0,w)\cdot(aw^d + \text{higher order terms}) + \sum_{j=1}^{d} r_j(0)\cdot w^{d-j}\,,$$

where $a \neq 0$. The left hand side contains no terms of order less than or greater than d in w, therefore $r_j(0) = 0$, and $h(0,0) \neq 0$. Put $q = h^{-1}$ and $P = w^d - \sum_{j=1}^{d} r_j(z)\cdot w^{d-j}$. □

As yet another application of Hironaka Division, we get

Theorem 3.10 (Implicit Function Theorem)**.** *Let $F(y,x) = (F_1(y,x),\dots,F_m(y,x))$, where $y = (y_1,\dots,y_n)$, $x = (x_1,\dots,x_m)$, and each $F_i(y,x) \in \mathbb{C}\{y,x\}$ (resp. $\mathbb{C}[[y,x]]$). Assume $F(0,0) = 0$, and $\frac{\partial F}{\partial x}(0,0)$ is invertible. Then there exists a unique $x = x(y) \in \mathbb{C}\{y\}^m$ (resp. $\mathbb{C}[[y]]^m$), such that $x(0) = 0$ and $F(y,x(y)) \equiv 0$.*

Proof. We can assume $\frac{\partial F}{\partial x}(0,0) = I$ is the identity matrix. Fix $1 \le j \le m$, and apply Theorem 3.6 (resp. 3.3) with $G_i = F_i$, $i = 1, \dots, m$, and $F = x_j$. Then $\mathrm{in}(G_i(0)) = x_i$, so $\Delta = \{0\}$, and we get

$$x_j = \sum_{i=1}^{m} g_{ij}(y,x) F_i(y,x) + h_j(y), \qquad j = 1, \dots, m\,.$$

Then $G = [g_{ij}]$ is invertible; in fact, $G(0,0) = I$. Indeed, put $(y,x) = (0,0)$ to see that $h_j(0) = 0$, and then differentiate with respect to x, at $(0,0)$, to get $I = G(0,0) \cdot \frac{\partial F}{\partial x}(0,0)$. We therefore have $F(y,x) = G(y,x)^{-1}(x - h(y))$, where $h(y) = (h_1(y), \dots, h_m(y))$. □

3.3 Diagram of initial exponents

Definition 3.11. Let I be an ideal in $A[[x]] = A[[x_1, \dots, x_m]]$, where $A = \mathbb{C}[[y]] = \mathbb{C}[[y_1, \dots, y_n]]$. The *diagram of initial exponents* of I, denoted as $\mathfrak{N}(I)$, is a subset of $\mathbb{N}^m$ defined as

$$\mathfrak{N}(I) \;=\; \{\exp(H) : H \in I \setminus \{0\}\}\,.$$

Remark 3.12. $\mathfrak{N}(I) + \mathbb{N}^m = \mathfrak{N}(I)$, since I is an ideal: $\exp(H \cdot x^\gamma) = \exp(H) + \gamma$ for $H \in A[[x]]$ and $\gamma \in \mathbb{N}^m$. If $I \subset \mathbb{C}\{y\}\{x\}$, then $\mathfrak{N}(I) = \mathfrak{N}(I \cdot \mathbb{C}[[y]][[x]])$, so we can assume that $I \subset \mathbb{C}[[y]][[x]]$.

Lemma 3.13. *Suppose $\mathfrak{N} \in \mathbb{N}^m$ and $\mathfrak{N} + \mathbb{N}^m = \mathfrak{N}$. Then there is a smallest finite subset V of $\mathfrak{N}$ such that $\mathfrak{N} = V + \mathbb{N}^m$. We call V the* vertices *of $\mathfrak{N}$.*

Proof. We proceed by induction on m. The case $m = 1$ is clear. For $m > 1$, let V denote the set of points $\beta \in \mathbb{N}^m$ such that

$$(\mathfrak{N} \setminus \{\beta\}) + \mathbb{N}^m \neq \mathfrak{N}\,.$$

It's easy to see that V is a set of vertices of $\mathfrak{N}$. To show that V is finite, it suffices to show that, for each i, the set

$$\{\beta_i : \beta = (\beta_1, \dots, \beta_i, \dots, \beta_m) \in V\}$$

is bounded. Consider, for example, β_m: By the inductive assumption, the projection of $\mathfrak{N}$ onto $\mathbb{N}^{m-1}$ has finitely many vertices $\alpha^1, \dots, \alpha^s$. Over every $\alpha^j = (\alpha_1^j, \dots, \alpha_{m-1}^j)$, there is some $\beta^j \in V$ in the sense that

$$\beta^j = (\alpha_1^j, \dots, \alpha_{m-1}^j, \beta_m^j)\,.$$

On the other hand, every other $\beta = (\beta_1, \dots, \beta_m) \in V$ must satisfy $\beta_m \le \max\{\beta_m^1, \dots, \beta_m^s\}$, for otherwise $\beta \in \beta^j + \mathbb{N}^m$ for some j. □

Corollary 3.14. *Let I be an ideal in $\mathbb{C}\{x\}$ or $\mathbb{C}[[x]]$, where $x = (x_1, \dots, x_m)$. Let $\mathfrak{N}(I)$ be the diagram of initial exponents of I, and let β^j, $j = 1, \dots, t$, denote the vertices of $\mathfrak{N}(I)$. Choose $G_1, \dots, G_t \in I$ so that $\exp(G_j) = \beta^j$, $j = 1, \dots, t$, and let $\{\Delta, \Delta_1, \dots, \Delta_t\}$ be the decomposition of $\mathbb{N}^m$ determined by the β^j, as before. Then:*

(1) $\mathfrak{N}(I) = \bigcup_j \Delta_j$, and the G_j generate I.

(2) There is a unique set of generators $F_1, \dots, F_t$ of I, such that, for each j, $\mathrm{in}(F_j) = x^{\beta^j}$ and $\mathrm{supp}(F_j - x^{\beta^j}) \subset \Delta$.

We call $F_1, \ldots, F_t$ the *standard basis* of I.

Proof. (1) Let $B = \mathbb{C}\{x\}$ or $\mathbb{C}[[x]]$ as the case may be. By Theorems 3.3 and 3.6, any $F \in B$ can be written as $F = \sum_i Q_i G_i + R$, where $\operatorname{supp}(R) \subset \Delta$. Then $F \in I$ iff $R \in I$. But, as $\operatorname{supp}(R) \subset \mathbb{N}^m \backslash \mathfrak{N}(I)$, $R \in I$ iff $R = 0$.

(2) For each $j = 1, \ldots, t$, divide x^{β^j} by $G_1, \ldots, G_t$: $x^{\beta^j} = \sum_i Q_i^j G_i + R^j$, where $\operatorname{supp}(R^j) \subset \Delta$. Put $F_j = x^{\beta^j} - R^j$. □

Corollary 3.15. *The rings* $\mathbb{C}[[x_1, \ldots, x_m]]$ *and* $\mathbb{C}\{x_1, \ldots, x_m\}$ *are noetherian.*

Proof. By Corollary 3.14, every ideal in $\mathbb{C}[[x]]$ or $\mathbb{C}\{x\}$ is finitely generated. □

Corollary 3.16. *Every ring of the form* $A[[x]]$ *or* $A\{x\}$*, where* $A = \mathbb{C}[[y]]/I$ *or* $A = \mathbb{C}\{y\}/I$*, is noetherian.*

Proof. Homomorphic images of noetherian rings are noetherian. □

Recall that Nakayama's Lemma implies that, if M is a finitely generated module over a local ring $(A, \mathfrak{m})$, then $M/\mathfrak{m}\cdot M$ is a finite-dimensional vector space over $A/\mathfrak{m}$. The converse is not true in general! Consider, e.g., $A = \mathbb{C}[y]_{(y)}$, $B = \mathbb{C}[y,x]_{(y,x)}$ and $M = B/(y^2 + x^2 + x^3)\cdot B$. Then $M/(y)\cdot M$ is a finite-dimensional $\mathbb{C}$-vector space, but M is not finitely generated as an A-module (**Exercise**). Nonetheless, the converse of Nakayama's Lemma *does* hold in the category of local analytic algebras. This also is a straightforward consequence of Hironaka's division theorem.

Theorem 3.17 (Weierstrass Finiteness Theorem)**.** *Let A be a local analytic $\mathbb{C}$-algebra and let I be an ideal in $A\{x\}$, where $x = (x_1, \ldots, x_m)$. Then $A\{x\}/I$ is a finitely generated A-module if and only if* $\dim_{\mathbb{C}}(\mathbb{C}\{x\}/I(0)) < \infty$.

Proof. If $A\{x\}/I$ is finitely generated over A, then $\dim_{\mathbb{C}}(\mathbb{C}\{x\}/I(0)) = \dim_{\mathbb{C}}(A\{x\}/I \otimes_A A/\mathfrak{m}_A) < \infty$, by Nakayama's Lemma. Conversely, suppose that $\dim_{\mathbb{C}}(\mathbb{C}\{x\}/I(0)) < \infty$. Let $G_1, \ldots, G_t$ be representatives of the vertices of the diagram $\mathfrak{N}(I(0))$; i.e., $G_1, \ldots, G_t \in I$ are such that $\exp(G_j(0)) = \beta^j$, where $\beta^1, \ldots, \beta^t$ are the vertices of $\mathfrak{N}(I(0))$. Let $\{\Delta, \Delta_1, \ldots, \Delta_t\}$ be the decomposition of $\mathbb{N}^m$ determined by the $\beta^1, \ldots, \beta^t$. Then, by Theorem 3.6, for every $F \in A\{x\}$, there are $Q_1, \ldots, Q_t, R \in A\{x\}$ such that $F = \sum_{j=1}^t Q_j G_j + R$ and $\operatorname{supp}(R) \subset \Delta$. On the other hand, the condition $\dim_{\mathbb{C}}(\mathbb{C}\{x\}/I(0)) < \infty$ means that Δ consists of finitely many points, say, $\gamma^1, \ldots, \gamma^s$. Thus every $R \in A\{x\}$ with $\operatorname{supp}(R) \subset \Delta$ is generated over A by the monomials $x^{\gamma^1}, \ldots, x^{\gamma^s}$. Hence, modulo I, every $F \in A\{x\}$ is generated over A by those finitely many monomials, which completes the proof. □

4 Rings of germs of holomorphic functions

4.1 Basic properties

Definition 4.1. Let M be an m-dimensional manifold, and let $a \in M$. Consider the set of holomorphic functions

$$\mathcal{H}ol_a = \{f \in \mathcal{O}(U) : U \text{ open neighbourhood of } a \text{ in } M\},$$

with the equivalence relation: $(V, f) \sim (W, g)$ iff $f|U \equiv g|U$ for some open $U \subset V \cap W$ containing a. The set $\mathcal{H}ol_a/\sim$ of germs at a of holomorphic functions in M forms a commutative ring, denoted $\mathcal{O}_{M,a}$ (or $\mathcal{O}_a(M)$), called the *ring of holomorphic germs at* a.

Remark 4.2. We list first a few simple observations (M is an m-dimensional manifold and $a \in M$ throughout):

(1) If $M = \mathbb{C}^m$, $\mathcal{O}_m := \mathcal{O}_{\mathbb{C}^m,0}$ is the ring of holomorphic germs at $0 \in \mathbb{C}^m$. (We identify $\mathcal{O}_0 = \mathcal{O}_{\mathbb{C}^0,0}$ with $\mathbb{C}$.) By abuse of notation, x_j will be used also to denote the germ $\{x \mapsto x_j\}_0 \in \mathcal{O}_m$, $j = 1, \dots, m$.

(2) $\mathcal{O}_{M,a}$ contains $\mathbb{C}$ as a subring (after identifying $\mathbb{C}$ with the germs at a of constant functions). $\mathcal{O}_{M,a}$ is thus a $\mathbb{C}$-vector space, and its ideals are $\mathbb{C}$-vector subspaces.

(3) If φ is a holomorphic mapping of an open neighbourhood of a into a manifold N, then the mapping

$$\varphi_a^* : \mathcal{O}_{N,\varphi(a)} \ni f_{\varphi(a)} \mapsto (f \circ \varphi)_a \in \mathcal{O}_{M,a}$$

is a ring homomorphism. Moreover, if φ is a biholomorphism of open neighbourhoods of a and $\varphi(a)$, then φ_a^* is an isomorphism (**Exercise**). In particular, $\mathcal{O}_{M,a} \cong \mathcal{O}_m$.

(4) Taylor expansion at $0 \in \mathbb{C}^m$ defines an isomorphism

$$\mathcal{O}_m \ni f_0 \mapsto \sum_{\beta \in \mathbb{N}^m} f_\beta x^\beta \in \mathbb{C}\{x\}, \qquad \text{where} \quad f_\beta = \frac{1}{\beta!}\frac{\partial^{|\beta|} f}{\partial x^\beta}(0).$$

We can thus identify the ring $\mathcal{O}_m$ (and in general $\mathcal{O}_{M,a}$) with $\mathbb{C}\{x\}$, where $x = (x_1, \dots, x_m)$.

(5) We will often identify $\mathcal{O}_k$, $k \le m$, with the subring of $\mathcal{O}_m$ of germs of functions independent of variables $x_{k+1}, \dots, x_m$, via the monomorphism $\mathcal{O}_k \ni f_0 \mapsto (f \circ \pi)_0 \in \mathcal{O}_m$, where $\pi(x_1, \dots, x_m) = (x_1, \dots, x_k)$ is a canonical projection. Hence, after the identifications,

$$\mathbb{C} = \mathcal{O}_0 \subset \mathcal{O}_1 \subset \dots \subset \mathcal{O}_m.$$

(6) The isomorphisms $\mathcal{O}_{M,a} \cong \mathcal{O}_m \cong \mathbb{C}\{x\}$ imply that, for every germ $f \in \mathcal{O}_{M,a}$, we have well-defined

$$f(a) := \tilde{f}(a) \qquad \text{and} \qquad \frac{\partial^{|\beta|} f}{\partial x^\beta} := \left(\frac{\partial^{|\beta|} \tilde{f}}{\partial x^\beta}\right)_a,$$

where $\tilde{f}$ is a representative of f at a.

(7) $f \in \mathcal{O}_{M,a}$ is invertible in $\mathcal{O}_{M,a}$ iff $f(a) \neq 0$. Hence $\mathcal{O}_{M,a}$ is a local ring, with the maximal ideal

$$\mathfrak{m} = \mathfrak{m}_a = \{f \in \mathcal{O}_{M,a} : f(a) = 0\}.$$

(8) The germs $x_j \in \mathcal{O}_m$ are irreducible ($j = 1, \dots, m$).
Indeed, for otherwise $x_j|_U = f(x)|_U \cdot g(x)|_U$ for some open neighbourhood U of 0 in $\mathbb{C}^m$ and $f, g \in \mathcal{O}(U)$ with $f(0) = g(0) = 0$. Then $1 = \frac{\partial}{\partial x_j}(x_j)(0) = \frac{\partial}{\partial x_j}(fg)(0) = 0$, by the product rule.

Proposition 4.3. *The ring $\mathcal{O}_{M,a}$ is noetherian.*

Proof. Follows from the isomorphism $\mathcal{O}_{M,a} \cong \mathbb{C}\{x\}$ and Corollary 3.15. □

Proposition 4.4. *The ring $\mathcal{O}_{M,a}$ is a regular local ring and $\dim \mathcal{O}_{M,a} = m$.*

Proof. Again, by Remark 4.2(4), it suffices to prove the statement for $\mathbb{C}\{x\}$, where $x = (x_1, \dots, x_m)$. We have $\dim \mathbb{C}\{x\} \geq m$, as

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \dots, x_m)$$

is a chain of prime ideals of length m. (That $(x_1, \dots, x_k)$ are prime follows from the isomorphism $\mathbb{C}\{x_1, \dots, x_m\}/(x_1, \dots, x_k) \cong \mathbb{C}\{x_1, \dots, x_{m-k}\}$ and Proposition 4.6 below.) On the other hand, a power series $f \in \mathbb{C}\{x\}$ is not invertible iff the constant term of f is zero. Thus, the unique maximal ideal $\mathfrak{m}$ of $\mathbb{C}\{x\}$ can be generated by m elements $x_1, \dots, x_m$, and so the embedding dimension $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2$ of $\mathbb{C}\{x\}$ equals m. This completes the proof, since a local ring $(R, \mathfrak{m}_R)$ is regular iff $\dim R = \mathrm{edim} R$ (and one always has $\dim R \leq \mathrm{edim} R$). □

Proposition 4.5. *Germs $f_1, \dots, f_m \in \mathcal{O}_{M,a}$ generate $\mathfrak{m}_a$ if and only if their differentials $d_a f_1, \dots, d_a f_m$ are linearly independent (over $\mathbb{C}$).*

Proof. It suffices to prove the claim for $\mathcal{O}_m$:
Consider a natural epimorphism $\varphi : \mathfrak{m} \ni f \mapsto d_0 f \in (T_0\mathbb{C}^m)^\times \cong \mathbb{C}^m$. Notice that $\ker \varphi = \mathfrak{m}^2$, and hence $\mathfrak{m}/\mathfrak{m}^2 \cong \mathbb{C}^m$. By Nakayama's Lemma and Proposition 4.4, $f_1, \dots, f_m$ generate $\mathfrak{m}$ iff their classes $\bar{f}_1, \dots, \bar{f}_m$ modulo $\mathfrak{m}^2$ generate $\mathfrak{m}/\mathfrak{m}^2$. Identifying $\bar{f}_j$ with $\varphi(f_j) = d_0 f_j$, we obtain the result. □

Proposition 4.6. *The ring $\mathcal{O}_{M,a}$ is an integral domain.*

Proof. Indeed, if $f, g \in \mathcal{O}_{M,a} \setminus \{0\}$, then there is a coordinate neighbourhood U of a, and representatives $\tilde{f}, \tilde{g} \in \mathcal{O}(U)$ of f and g respectively, such that $\tilde{f} \not\equiv 0$ and $\tilde{g} \not\equiv 0$ on U. Then $\{\tilde{f}\tilde{g} = 0\} = \{\tilde{f} = 0\} \cup \{\tilde{g} = 0\}$, as a proper analytic subset, is nowhere dense in U, by Theorem 2.7. Hence $\tilde{f}\tilde{g} \not\equiv 0$, and thus $fg = (\tilde{f}\tilde{g})_a \neq 0$. □

Proposition 4.7. *Every non-constant germ $f \in \mathcal{O}_m$ with $f(0) = 0$ is regular (after a linear change of coordinates, at worst) with respect to some x_j.*

Proof. We will show that a non-constant $f \in \mathfrak{m}$ is x_m-regular after a suitable linear change of coordinates.
Write $f(x) = \sum_{\nu \in \mathbb{N}} f_\nu(x)$, where $f_\nu(x) = \sum_{|\beta| = \nu} f_\beta x^\beta$ is a form of degree ν. The assumptions $f \in \mathfrak{m}$ and $f \not\equiv \mathrm{const}$ imply that there exists $r \in \mathbb{N}_+$ such that $f_r(x) \not\equiv 0$ and $f_k(x) \equiv 0$ for all $k < r$. Then $f = \sum_{\nu=r}^{\infty} f_\nu(x)$. Let $U = \epsilon\Delta^m$ be a polydisc on which f is convergent. Then $A := \{f_r = 0\}$ is a proper analytic subset of U. Pick $z \in U \setminus A$; after a linear change of coordinates at $0 \in \mathbb{C}^m$, we may assume that $z = (0, \dots, 0, 1)$. In the new coordinates,

$$f(0, \dots, 0, x_m) = f(z \cdot x_m) = \sum_{\nu=r}^{\infty} f_\nu(z) x_m^\nu$$

is regular of order r in x_m. □

Remark 4.8. In fact one can show more: Given a finite collection of non-constant germs $f_1, \dots, f_t \in \mathcal{O}_m$, there is a nowhere dense closed subset S of the space of linear isomorphisms $\mathcal{L}(\mathbb{C}^m, \mathbb{C}^m)$ such that, for every linear change of coordinates $\varphi \notin S$, $f_1, \dots, f_t$ are all regular in the same variable x_j $(1 \leq j \leq m)$. **Exercise**.

Proposition 4.9. *The ring $\mathcal{O}_{M,a}$ is a unique factorization domain.*

Proof. It suffices to consider $\mathcal{O}_m$. Induction on m: If $m = 0$, then $\mathcal{O}_m = \mathbb{C}$ is *UFD* as a field. Suppose then that $\mathcal{O}_{m-1}$ is *UFD*. By the Gauss Lemma, so is $\mathcal{O}_{m-1}[x_m]$. Suppose now that an irreducible germ $f \in \mathcal{O}_m$ divides the product $gh \in \mathcal{O}_m$. We want to show that f divides one of the factors. By Remark 4.8 above, we may without loss of generality assume that all f, g and h are x_m-regular. Let $F, G, H \in \mathcal{O}_{m-1}[x_m]$ and $r, s, t \in \mathcal{O}_m \setminus \mathfrak{m}$ be such that $f = rF$, $g = sG$, and $h = tH$ (Weierstass Preparation). Then $f|gh$ implies that $F|GH$ in $\mathcal{O}_{m-1}[x_m]$, and F is irreducible in $\mathcal{O}_{m-1}[x_m]$ (being an associate of f). Now, since $\mathcal{O}_{m-1}[x_m]$ is a *UFD*, it follows that $F|G$ or $F|H$, hence $f|g$ or $f|h$ respectively. □

4.2 Analytic germs

Definition 4.10. Let M be an m-dimensional manifold, and let $a \in M$. Given a germ $f \in \mathcal{O}_{M,a}$, we define a set-germ

$$\mathcal{V}(f) := \{x \in M : \tilde{f}(x) = 0\}_a\,, \quad \text{where } \tilde{f} \text{ is a representative of } f \text{ at } a\,,$$

which we call the *zero set germ* of f.

Remark 4.11.

(1) The *zero set germ* definition is independent of the choice of representative.

(2) If f and g are associates in $\mathcal{O}_{M,a}$, then $\mathcal{V}(f) = \mathcal{V}(g)$.

(3) $\mathcal{V}(f_1 \cdots f_k) = \mathcal{V}(f_1) \cup \cdots \cup \mathcal{V}(f_k)$ for $f_1, \dots, f_k \in \mathcal{O}_{M,a}$.

(4) One defines $\mathcal{V}(f_1, \dots, f_k) := \mathcal{V}(f_1) \cap \cdots \cap \mathcal{V}(f_k)$. We have $\mathcal{V}(f_1, \dots, f_k) = \{\tilde{f}_1 = \cdots = \tilde{f}_k = 0\}_a$, where $\tilde{f}_1, \dots, \tilde{f}_k$ are arbitrary representatives at a of $f_1, \dots, f_k$ respectively.

(5) Given an ideal I in $\mathcal{O}_{M,a}$, one defines the zero set germ of I as $\mathcal{V}(I) = \mathcal{V}(f_1, \dots, f_k)$ $(= \mathcal{V}(f_1) \cap \cdots \cap \mathcal{V}(f_k))$, where $f_1, \dots, f_k$ generate I. The definition is independent of the choice of generators.

(6) For ideals $I_1, \dots, I_k$ in $\mathcal{O}_{M,a}$, we have

$$\mathcal{V}(I_1 + \cdots + I_k) = \mathcal{V}(I_1) \cap \cdots \cap \mathcal{V}(I_k)\,.$$

(7) Given ideals I and J in $\mathcal{O}_{M,a}$,

$$I \subset J \;\Rightarrow\; \mathcal{V}(I) \supset \mathcal{V}(J)\,.$$

In particular, $\mathcal{V}(I) \subset \mathcal{V}(f)$ for any $f \in I$. Hence (**Exercise**), for $I_1, \dots, I_k$,

$$\mathcal{V}(I_1 \cap \cdots \cap I_k) = \mathcal{V}(I_1) \cup \cdots \cup \mathcal{V}(I_k)\,.$$

(8) For any ideal I, we have

$$\mathcal{V}(I) = \mathcal{V}(\mathrm{rad} I)$$

(**Exercise** - combine properties 3, 4 and 7 above).

Definition 4.12. A germ A at a is called an *analytic germ*, when it is a germ of a locally analytic set through a; i.e., A has a representative $\tilde{A}$ analytic in some neighbourhood of a (and $a \in \tilde{A}$). Equivalently, $A = \mathcal{V}(h_1, \dots, h_s) = \mathcal{V}((h_1, \dots, h_s))$ for some $h_1, \dots, h_s \in \mathcal{O}_{M,a}$. The ideal $(h_1, \dots, h_s)$ is then called a *defining ideal* of A.

An analytic germ A is called *smooth* (or *non-singular*) when it is a germ of a submanifold of M through a. Otherwise, A is called *singular*.

Definition 4.13. Let A be an analytic germ at a. We say that $f \in \mathcal{O}_{M,a}$ *vanishes on* A when some representative of f at a vanishes on some representative of A at a. Equivalently, $A \subset \mathcal{V}(f)$. The set

$$\mathfrak{J}(A) = \{f \in \mathcal{O}_{M,a} : f \text{ vanishes on } A\} = \{f \in \mathcal{O}_{M,a} : A \subset \mathcal{V}(f)\}$$

forms an ideal in $\mathcal{O}_{M,a}$, which we call the *full ideal of* A (or the *ideal of* A, for short).

Remark 4.14.

(1) $A \subset B \ \Rightarrow\ \mathfrak{J}(A) \supset \mathfrak{J}(B)$.

(2) $\mathfrak{J}(A_1 \cup \dots \cup A_k) = \mathfrak{J}(A_1) \cap \dots \cap \mathfrak{J}(A_k)$.

(3) For any analytic germ A we have

$$\mathcal{V}(\mathfrak{J}(A)) = A\,.$$

Indeed, if $\mathfrak{J}(A)$ is generated by $g_1, \dots, g_s$, then $\mathcal{V}(\mathfrak{J}(A)) = \bigcap_1^s \mathcal{V}(g_i) \supset A$. On the other hand, $A = \mathcal{V}(h_1, \dots, h_t) = \mathcal{V}(h_1) \cap \dots \cap \mathcal{V}(h_t)$ for some $h_j \in \mathcal{O}_{M,a}$. Then $h_j \in \mathfrak{J}(A)$, hence $\mathcal{V}(h_j) \supset \mathcal{V}(\mathfrak{J}(A))$ for $j = 1, \dots, t$, and thus $A \supset \mathcal{V}(\mathfrak{J}(A))$.

(4) Now, by properties 1, 3, and Remark 4.11(7), for every pair of germs A and B,

$$A \subset B \ \Leftrightarrow\ \mathfrak{J}(A) \supset \mathfrak{J}(B) \qquad \text{and} \qquad A = B \ \Leftrightarrow\ \mathfrak{J}(A) = \mathfrak{J}(B)\,.$$

Hence, by noetherianity of $\mathcal{O}_{M,a}$:

Proposition 4.15. *Every decreasing sequence of analytic germs is stationary.*

Definition 4.16. An analytic germ A is called *reducible* when $A = A_1 \cup A_2$ for some proper analytic subgerms A_1 and A_2. A germ which is not reducible is called *irreducible*.

Remark 4.17. If A is an irreducible analytic germ, then, for any collection $B_1, \dots, B_k$ of analytic germs, we have (**Exercise**)

$$A \subset B_1 \cup \dots \cup B_k \quad \Rightarrow \quad A \subset B_j \text{ for some } j\,.$$

Proposition 4.18. *Every analytic germ is the union of a unique finite collection of irreducible analytic germs* $\{A_j\}$*, satisfying* $A_i \not\subset \bigcup_{j \neq i} A_j$*, which we call its* irreducible components.

Proof. **Exercise**. □

Remark 4.19. The following two properties are not difficult to verify (**Exercise**):

(1) Every smooth analytic germ is irreducible.

(2) An analytic germ A is irreducible iff its ideal $\mathfrak{J}(A)$ is prime.

Definition 4.20. An analytic set $X \subset M$ is called *locally irreducible* when its germ X_ξ at every point $\xi \in X$ is irreducible.

5 Proper projections

5.1 Proper projections

Recall that a mapping $\varphi : X \to Y$ of topological spaces is called *proper* when $\varphi^{-1}(K)$ is compact for every compact $K \subset Y$. We list here a few simple observations regarding proper projections. Throughout this section M is an m-dimensional manifold, k is a positive integer, X is a closed subset of $M \times \mathbb{C}^k$, and

$$\pi : M \times \mathbb{C}^k \ni (y, x) \ \mapsto \ y \in M$$

is the canonical projection.

Lemma 5.1. *The following conditions are equivalent:*

(i) $\pi|_X : X \to M$ *is a proper projection.*

(ii) For every $y_0 \in M$ *there are a coordinate neighbourhood* U *of* y_0 *in* M*, and* $R > 0$*, such that*

$$y \in U, \ (y, x) \in X \quad \Longrightarrow \quad x \in R\Delta^k \,.$$

Proof. **Exercise**. □

Lemma 5.2. *Suppose* $Z \subset M \times \mathbb{C}^k$ *is such that the restriction* $\pi|_Z : Z \to M$ *is proper. Then:*

(i) Z *is closed.*

(ii) $\pi|_Z : Z \to M$ *is a closed mapping.*

Proof. (i): Suppose $(z_n)_1^\infty \subset Z$ is a convergent sequence, with $z_n \xrightarrow[n\to\infty]{} z_0 \in M \times \mathbb{C}^k$. Then $K = \{z_n : n \in \mathbb{N}\}$ is compact. By continuity and properness of $\pi|_Z$, $\pi^{-1}(\pi(K)) \cap Z = (\pi|_Z)^{-1}(\pi(K))$ is compact as well, and hence $z_0 \in Z$, as $(z_n)_1^\infty \subset \pi^{-1}(\pi(K)) \cap Z$.

(ii): Let $F \neq \varnothing$ be a closed subset of Z, and let $(y_n)_1^\infty \subset \pi(F)$ be a convergent sequence, with $y_n \xrightarrow[n\to\infty]{} y_0 \in M$. Put $K = \{y_n : n \in \mathbb{N}\}$. Then $(\pi|_Z)^{-1}(K) \cap Z$ is compact, as the intersection of a compact and a closed set, and there exists $(z_n)_1^\infty \subset (\pi|_Z)^{-1}(K) \cap Z$ such that $\pi(z_n) = y_n$. Now, $(z_n)_1^\infty$ contains a subsequence convergent to $z_0 \in (\pi|_Z)^{-1}(K) \cap Z$, hence $y_0 = \pi(z_0) \in \pi(F)$, as required. □

Lemma 5.3. *Suppose* $\pi|_X : X \to M$ *is proper,* $y_0 \in M$*, and* $r > 0$ *are such that*

$$(\pi|_X)^{-1}(y_0) \subset \{y_0\} \times r\Delta^k \,.$$

Then there is a coordinate neighbourhood U *of* y_0 *in* M *for which*

$$(\pi|_X)^{-1}(U) \ = \ X \cap (U \times r\Delta^k) \,.$$

Proof. The set $Z = X \setminus (M \times r\Delta^k)$ is closed. Hence $\pi(Z)$ is closed in M, by Lemma 5.2, and $y_0 \notin \pi(Z)$. Let U be a coordinate neighbourhood of y_0 in M for which $U \subset M \setminus \pi(Z)$. Now, $X = (X \cap (M \times r\Delta^k)) \cup Z$, and hence

$$(\pi|_X)^{-1}(U) = (U \times \mathbb{C}^k) \cap X = \big((U \times \mathbb{C}^k) \cap X \cap (M \times r\Delta^k)\big) \cup \big((U \times \mathbb{C}^k) \cap Z\big) = \big((U \times r\Delta^k) \cap X\big) \cup \varnothing = (U \times r\Delta^k) \cap X \,.$$

□

Let now $f : \mathbb{C}^k \to \mathbb{C}^l$, $\Phi_f : M \times \mathbb{C}^k \ni (y, x) \mapsto (y, f(x)) \in M \times \mathbb{C}^l$, and let $\tilde{\pi} : M \times \mathbb{C}^l \to M$ be the canonical projection.

Lemma 5.4. *Suppose $\pi|_X : X \to M$ is proper, and $f : \mathbb{C}^k \to \mathbb{C}^l$ is continuous. Then:*

(i) $(\Phi_f|_X) : X \to M \times \mathbb{C}^l$ *is proper.*

(ii) $X_f := \Phi_f(X)$ *is a closed subset of* $M \times \mathbb{C}^l$.

(iii) $\tilde{\pi}|_{X_f} : X_f \to M$ *is proper.*

Proof. **Exercise**. □

Lemma 5.5. *Suppose M is connected. Let $a_0, \dots, a_d : M \to \mathbb{C}$ be continuous, with $a_0 \not\equiv 0$, and let*

$$X = \{(y, x) \in M \times \mathbb{C} : a_0(y)x^d + a_1(y)x^{d-1} + \dots + a_{d-1}(y)x + a_d(y) = 0\}.$$

Then $\pi|_X : X \to M$ is proper iff $a_0(y) \neq 0$ for every $y \in M$.

Proof. "$\Leftarrow$": Let $K \subset M$ be compact, and let

$$R = 2 \cdot \max_{\substack{y \in K \\ j=1,\dots,d}} \left| \frac{a_j(y)}{a_0(y)} \right|^{1/j}.$$

Then all the roots of $a_0(y)x^d + \dots + a_d(y)$ lie within $R\Delta$ for all $y \in K$. Hence $(\pi|_X)^{-1}(K) = X \cap (K \times R\Delta)$ is compact.

"$\Rightarrow$": Put $S = a_0^{-1}(0)$ and suppose that $S \neq \varnothing$. Choose $y_0 \in \partial S$. Then $a_0(y_0) = 0$ (by continuity), and there exists $(y_n)_1^\infty \subset M \setminus S$ such that $y_n \xrightarrow[n \to \infty]{} y_0$. The set $K = \{y_n : n \in \mathbb{N}\}$ is compact, so by Lemma 5.1, there exists $R > 0$ for which

$$y \in K,\ (y, x) \in X \quad \Rightarrow \quad |x| \leq R.$$

Then, for every $n \geq 1$, $\left| \frac{a_j(y_n)}{a_0(y_n)} \right| \leq \binom{d}{j} R^j$ (by Viéte's formulas), hence $|a_j(y_n)| \leq \binom{d}{j} R^j |a_0(y_n)|$, and consequently $a_j(y_0) = 0$ for $j = 0, 1, \dots, d$. Therefore X contains the entire line $\{y_0\} \times \mathbb{C}$, contradicting the properness of $\pi|_X$. It follows that $\partial S = \varnothing$. But S is a proper closed subset of a connected manifold, so $S = \varnothing$. □

5.2 Resultant and discriminant

We recall first the notion of resultant. (Here w stands for a single variable.)

Definition 5.6. Let A be an integral domain. Given two polynomials $P = a_0 w^p + \dots + a_{p-1} w + a_p$ and $Q = b_0 w^q + \dots + b_{q-1} w + b_q$ in $A[w]$, one defines the *resultant* of P and Q as the determinant

$$R(P, Q) = \begin{vmatrix} a_0 & a_1 & \dots & a_p & & & \\ & a_0 & a_1 & \dots & a_p & & \\ & & \ddots & & & \ddots & \\ & & & a_0 & a_1 & \dots & a_p \\ b_0 & b_1 & \dots\dots & b_q & & & \\ & \ddots & & & & \ddots & \\ & & b_0 & b_1 & \dots\dots & & b_q \end{vmatrix}$$

where the number of "a_i-rows" is equal to q, the number of "b_j-rows" is equal to p, and the blank spaces are filled with zeros.

We have the following very useful result.

Theorem 5.7. *Let A be an integral domain, and let $P, Q \in A[w]$ be nonconstant polynomials. Then there exist polynomials $F, G \in A[w]$ such that*

$$FP + GQ = R(P,Q)\,, \tag{5.1}$$

with $\deg F < \deg Q$ and $\deg G < \deg P$. In particular, if P and Q have a common factor (of positive degree), then $R(P,Q) = 0$ in A. If moreover A is a UFD, then $R(P,Q) = 0$ implies that P and Q have a common factor.

Proof. Let $P = a_0w^p + a_1w^{p-1} + \dots + a_p$ and $Q = b_0w^q + b_1w^{q-1} + \dots + b_q$. Multiply P and Q by the consecutive powers of w and consider the following system of $p+q$ equations

$$\begin{array}{rlllllll}
w^{q-1}P = & a_0w^{p+q-1} + & a_1w^{p+q-2} & + \dots & + & a_pw^{q-1} & & \\
w^{q-2}P = & & a_0w^{p+q-2} & + \dots & & + & a_pw^{q-2} & \\
\multicolumn{8}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\
P = & & & a_0w^p & + & \dots\dots\dots\dots & & + a_p \\
w^{p-1}Q = & b_0w^{p+q-1} + & b_1w^{p+q-2} & + \dots & + & b_qw^{p-1} & & \\
w^{p-2}Q = & & b_0w^{p+q-2} & + \dots & & + & b_qw^{p-2} & \\
\multicolumn{8}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\
Q = & & & b_0w^q & + & \dots\dots\dots\dots & & + b_q
\end{array}$$

Let C be the column vector on the left hand side, and let $C_0, \dots, C_{p+q}$ be the column vectors of the coefficients. Then the above system of equations can be written as

$$C = w^{p+q-1} \cdot C_0 + w^{p+q-2} \cdot C_1 + \dots + w \cdot C_{p+q-1} + 1 \cdot C_{p+q}\,.$$

Now, treating the $w^{p+q-1}, \dots, w, 1$ as independent variables and applying Cramer's Rule to the last variable (which is 1), we get

$$1 \cdot \det(C_0, \dots, C_{p+q}) = \det(C_0, \dots, C_{p+q-1}, C)\,.$$

Notice that $\det(C_0, \dots, C_{p+q}) = R(P,Q)$ and the right hand side can be written as $FP + GQ$ with F and G in $A[w]$, of degrees $q-1$ and $p-1$ respectively (by expanding the matrix $(C_0, \dots, C_{p+q-1}, C)$ according to the last column).

Next, suppose that P and Q have a common factor. Then the polynomial on the left hand side of (5.1) has a root. On the other hand, the right hand side of (5.1) is a constant polynomial, so it has a root iff it is identically zero.

Finally, suppose that A is a *UFD*, $R(P,Q) = 0$, and P and Q have no common factor of positive degree. Let F and G be the polynomials from (5.1). Then $FP = -GQ$, hence every irreducible factor of Q divides FP. Since P and Q are relatively prime, it follows that Q divides F. But this is impossible, as $\deg F < \deg Q$. □

Definition 5.8. Let A be *UFD* and let $P \in A[w]$ be a monic polynomial of degree d. One defines the *discriminant* of P, denoted $D(P)$, as

$$D(P) = (-1)^{\binom{d}{2}} \cdot R(P, \frac{\partial P}{\partial w})\,.$$

Notice that $D(P) \in A$. In our considerations, we will be interested in the case when $A = \mathcal{O}_{M,a}$ for some manifold M (which is a *UFD*, by Proposition 4.9). We then have the following

Lemma 5.9. *Let M be a manifold, $P = w^d + a_1 w^{d-1} + \cdots + a_d \in \mathcal{O}(M)[w]$, and let $D = D(P)$. Then*

$$D(z) = \prod_{i<j}(w_i - w_j)^2 = (-1)^{\binom{d}{2}} \prod_{j=1}^{d} \frac{\partial P}{\partial w}(z, w_j)$$

for $z \in M$, where $w_1, \dots, w_d$ are the roots of the polynomial $\{w \mapsto P(z, w)\}$.

Proof. **Exercise** [Hint: Argue pointwise. Notice that, for a fixed $z \in M$, $P(z, \cdot) \in \mathbb{C}[w]$ is a product of precisely d linear factors.] □

In general, we have:

Proposition 5.10. *Let A be a UFD of characteristic zero, and let $P \in A[w]$ be monic. The following conditions are equivalent:*

(i) P is divisible by Q^2 for some $Q \in A[w]$ of positive degree

(ii) P and $\partial P/\partial w$ have a common factor of positive degree

(iii) Discriminant $D(P)$ is zero.

Proof. See, e.g., [Ło, A.6.3]. □

Remark 5.11.

(1) Given A and monic $P \in A[w]$ as above, there are unique (distinct) monic irreducible polynomials $P_1, \dots, P_s$ and positive integers $m_1, \dots, m_s$ such that $P = P_1^{m_1} \dots P_s^{m_s}$ in $A[w]$. We then define

$$\mathrm{red}P = P_1 \dots P_s\,.$$

It follows from Proposition 5.10 that $D(\mathrm{red}P) \neq 0$.

(2) For a distinguished germ $P \in \mathcal{O}_{M,a}[w]$, we put

$$\mathrm{red}P = (\mathrm{red}\widetilde{P})_a\,,$$

where $\widetilde{P} \in \mathcal{O}(U)[w]$ is a monic representative of P at a. Since, for every $z \in U$, the roots of $\widetilde{P}$ and $\mathrm{red}\widetilde{P}$ are the same, it follows that $\{\widetilde{P} = 0\} = \{\mathrm{red}\widetilde{P} = 0\}$ as subsets of $U \times \mathbb{C}$. In particular, if $P \in \mathcal{O}_{M,a}[w]$ is distinguished, then so is $\mathrm{red}P$.

5.3 Remert Proper Mapping Theorem

Let, again, M be an m-dimensional manifold, $k \geq 1$, and let $\pi : M \times \mathbb{C}^k \ni (z, x) \mapsto z \in M$ be the projection.

Theorem 5.12 (Remmert Proper Mapping Theorem). *If X is analytic in $M \times \mathbb{C}^k$, and $\pi|_X : X \to M$ is proper, then $\pi(X)$ is analytic in M.*

Proof. Induction on k: The case $k = 1$ follows from Lemmas 5.13 and 5.15 below, so let's suppose the theorem holds for $k - 1 \geq 1$ and X is analytic in $M \times \mathbb{C}^k$ with proper projection onto M.
Let $f(x_1, \dots, x_{k-1}, x_k) = (x_1, \dots, x_{k-1})$, $\Phi_f : M \times \mathbb{C}^k \ni (z, x) \mapsto (z, f(x)) \in M \times \mathbb{C}^{k-1}$, and let $\pi_{k-1} : M \times \mathbb{C}^{k-1} \to M$ be the projection. Then Φ_f is the projection $(M \times \mathbb{C}^{k-1}) \times \mathbb{C} \to M \times \mathbb{C}^{k-1}$, and $\Phi_f|_X$ is proper (by Lemma 5.4), hence $X_f = \Phi_f(X)$ is analytic in $M \times \mathbb{C}^{k-1}$. Now, $\pi_{k-1}|_{X_f} : X_f \to M$ is again proper, by Lemma 5.4, hence $\pi(X) = \pi_{k-1}(X_f)$ is analytic in M, by the inductive hypothesis. □

To complete the proof of Theorem 5.12, it remains to show the following two lemmas.

Lemma 5.13. *Suppose X is analytic in $M \times \mathbb{C}$, the projection $\pi|_X : X \to M$ is proper, and $z_0 \in \pi(X)$. Then there exist a coordinate neighbourhood U of z_0 in M, positive integers d and s, and polynomials $W_0, \dots, W_s \in \mathcal{O}(U)[w]$ such that:*

(i) W_0 *is monic of degree d in w*

(ii) $\deg W_j < \deg W_0$ *for all $1 \leq j \leq s$, and the coefficients of $W_1, \dots, W_s$ vanish at* 0

(iii) $X \cap (U \times \mathbb{C}) = \bigcap_{j=0}^{s} W_j^{-1}(0)$.

Proof. The problem being local, we can assume that $M = \mathbb{C}^m$. By Example 2.5(4), $(\pi|_X)^{-1}(z_0)$ is finite; say, $(\pi|_X)^{-1}(z_0) = \{(z_0, w_1), \dots, (z_0, w_p)\}$. Let U be a connected open neighbourhood of $z_0 \in \mathbb{C}^m$, and let $r > 0$ be such that $B_r(w_i) \cap B_r(w_j) = \emptyset$ for $i \neq j$, and, for all $i = 1, \dots, p$,

$$X \cap (U \times B_r(w_i)) = \{h_{i1} = \dots = h_{iq_i} = 0\} \qquad \text{for some} \quad h_{ij} \in \mathcal{O}(U \times B_r(w_i))\,,$$

where $B_r(w)$ denotes an open ball in $\mathbb{C}$ centered at w with radius r. Since $X \cap (U \times B_r(w_i)) \cap (\{z_0\} \times \mathbb{C}) = \{(z_0, w_i)\}$, at least one of the functions $h_{ij}(z_0, \cdot)$ has an isolated zero at w_i; say, $h_{i1}(z_0, \cdot)$ does so. Then the germs $(h_{11})_{(z_0,w_1)}, \dots, (h_{p1})_{(z_0,w_p)}$ are regular in w, so after shrinking U and r if necessary, there are (by Weierstrass Preparation at the (z_0, w_i)) monic polynomials $W_0^1, \dots, W_0^p \in \mathcal{O}(U)[w]$ with

$$h_{i1}^{-1}(0) \cap (U \times B_r(w_i)) \;=\; (W_0^i)^{-1}(0) \cap (U \times B_r(w_i)) \qquad \text{for } i = 1, \dots, p\,.$$

Now, by Weierstrass Division, for all $i = 1, \dots, p$ and $j = 1, \dots, q_i$ we can divide

$$h_{ij} = g_{ij} W_0^i + W_j^i, \quad \text{where} \quad W_j^i \in \mathcal{O}(U)[w] \quad \text{and} \quad \deg W_j^i < \deg W_0^i\,,$$

again, after shrinking U and r if necessary. Then $X \cap (U \times B_r(w_i)) = \{W_0^i = \dots = W_{q_i}^i = 0\}$, and we obtain the result by putting

$$W_0 = W_0^1 \cdots W_0^p\,,$$

and $W_1, \dots, W_s$ to be the remaining products of the form

$$W_{j_1}^1 \cdots W_{j_p}^p\,, \quad \text{where} \quad j_1 + \dots + j_p \geq 1\,.$$

□

Corollary 5.14. *Under the assumptions of Lemma 5.13, there exist a coordinate neighbourhood U of z_0 in M, and integer $d \geq 1$, such that*

$$\#\left(\pi|_X\right)^{-1}(z) \leq d \qquad \text{for all } z \in U\,.$$

Proof. Indeed, if W_0 is the monic polynomial from Lemma 5.13, then $d = \deg W_0$ will do, as $X \cap (U \times \mathbb{C}) \subset W_0^{-1}(0)$. □

Lemma 5.15. *Suppose X is analytic in $M \times \mathbb{C}$, the projection $\pi|_X : X \to M$ is proper, and $z_0 \in \pi(X)$. Then there exists a coordinate neighbourhood U of z_0 in M, such that $\pi(X \cap (U \times \mathbb{C}))$ is analytic in U.*

Proof. As in the proof of Lemma 5.13, we can assume that $M = \mathbb{C}^m$. Let U, and $W_0, W_1, \dots, W_s \in \mathcal{O}(U)[w]$ be as in Lemma 5.13. Consider the polynomial

$$S(z, w, \lambda_1, \dots, \lambda_s) = W_0(z, w) + \lambda_1 W_1(z, w) + \dots + \lambda_s W_s(z, w)$$

in $w, \lambda_1, \dots, \lambda_s$, monic with respect to w, and let

$$R(z, \lambda_1, \dots, \lambda_s) = R(W_0(z, w), S(z, w, \lambda_1, \dots, \lambda_s)) \ \in \mathcal{O}(U)[\lambda_1, \dots, \lambda_s]$$

be the resultant of W_0 and S (as polynomials in w). We claim that

$$z \in \pi(X \cap (U \times \mathbb{C})) \quad \Leftrightarrow \quad R(z, \lambda_1, \dots, \lambda_s) = 0 \text{ for all } \lambda_1, \dots \lambda_s \in \mathbb{C}\,. \tag{5.2}$$

"$\Rightarrow$": If $z' \in \pi(X \cap (U \times \mathbb{C}))$, then $(z', w') \in X$ for some $w' \in \mathbb{C}$, hence both W_0 and S vanish at (z', w') for arbitrary λ_j's. Thus, by Theorem 5.7, the resultant $R(z', \lambda_1, \dots, \lambda_s)$ vanishes as well.
"$\Leftarrow$": Suppose that $R(z', \lambda_1, \dots, \lambda_s) = 0$ for all $\lambda_1, \dots, \lambda_s \in \mathbb{C}$. Notice that, for a fixed z', $W(z', w)$ and $S(z', w, \lambda_1, \dots, \lambda_s)$ are (with respect to w) polynomials with coefficients in $\mathbb{C}[\lambda_1, \dots, \lambda_s]$, which is a *UFD*. Hence, by Theorem 5.7 again, $R(W_0(z', \cdot), S(z', \cdot, \lambda_1, \dots, \lambda_s)) = 0$ for all $\lambda_1, \dots, \lambda_s \in \mathbb{C}$ implies that $W_0(z', \cdot)$ and $S(z', \cdot, \lambda_1, \dots, \lambda_s)$ have a common factor for all $\lambda_1, \dots, \lambda_s \in \mathbb{C}$.

Since $W_0(z', \cdot)$ vanishes only at finitely many points, say $w_1, \dots, w_d$, then, for all $\lambda_1, \dots, \lambda_s \in \mathbb{C}$,

$$(\lambda_1 W_1(z', w_1) + \dots + \lambda_s W_s(z', w_1)) \cdot \ \cdots \ \cdot (\lambda_1 W_1(z', w_d) + \dots + \lambda_s W_s(z', w_d)) = 0\,.$$

The ring $\mathbb{C}[\lambda_1, \dots, \lambda_s]$ being an integral domain, there exists w_j for which

$$\lambda_1 W_1(z', w_j) + \dots + \lambda_s W_s(z', w_j) = 0\,,$$

for all $\lambda_1, \dots, \lambda_s \in \mathbb{C}$. Then $W_1(z', w_j) = \dots = W_s(z', w_j) = 0$, and hence $z' \in \pi(X \cap (U \times \mathbb{C}))$.

Now, by the construction of resultant,

$$R(z, \lambda_1, \dots, \lambda_s) = \sum_{\substack{\alpha \in \mathbb{N}^s \\ |\alpha| \leq D}} R_\alpha(z) \lambda^\alpha\,, \quad \text{where} \quad R_\alpha \in \mathcal{O}(U)\,.$$

It thus follows from (5.2) that $\pi(X \cap (U \times \mathbb{C})) = \bigcap_{|\alpha| \leq D} R_\alpha^{-1}(0)$, which completes the proof. □

5.4 Dimension of a proper projection

We complete this section with a theorem stating that a proper projection of an analytic set preserves the dimension. First, we prove "the obvious" inequality.

Lemma 5.16. *Suppose X is analytic in $M \times \mathbb{C}$, $\pi|_X : X \to M$ is proper, and $\pi(X) = M$. Then $\dim X \geq \dim M$.*

Proof. Let $z_0 \in M$ and let U be its coordinate neighbourhood, such that $X \cap (U \times \mathbb{C}) = \bigcap_{i=0}^{s} W_i^{-1}(0)$, as in Lemma 5.13, where W_0 is a monic polynomial in w. By Remark 5.11, we may assume that W_0 is reduced in $\mathcal{O}(U)[w]$, and hence the discriminant $\delta := D(W_0) \in \mathcal{O}(U)$ is not identically 0 on U. Let $Z = (U \times \mathbb{C}) \cap W_0^{-1}(0)$, and let $d = \max\{\#(\pi|_Z)^{-1}(z) : z \in U\}$ $(= \deg W_0)$. By replacing z_0 with another point of U, we may assume that $(\pi|_Z)^{-1}(z_0) = \{(z_0, w_1), \dots, (z_0, w_d)\}$. Then, by Lemma 5.9, $\delta(z_0) \neq 0$ and hence, for a sufficiently small neighbourhood V of z_0, $\delta(z) \neq 0$ for all $z \in V$. Therefore, for a small enough $r > 0$, the set $Z \cap (\bigcup_1^d (V \times B_r(w_j)))$ is a disjoint union of graphs of d holomorphic functions $h_1, \dots, h_d \in \mathcal{O}(V)$.

Indeed, W_0 being monic in w, it follows from Lemmas 5.1 and 5.5 that the germ $(W_0)_{(z_0,w_j)}$ is regular in $w - w_j$ $(j = 1, \dots, d)$. Then, by Weierstrass Preparation, $W_0 = W_j' \cdot W_j''$ within sufficiently small $V \times B_r(w_j)$, where $W_j'' \neq 0$ on $V \times B_r(w_j)$ and $(W_j')_{(z_0,w_j)}$ is distinguished in $w - w_j$. Since $W_0(z, \cdot)$ has only simple roots for $z \in V$, it follows that $\deg W_j' = 1$ (after replacing W_j' by $\mathrm{red}(W_j')$ if necessary). Therefore $W_j'(z, w) = w + a_{j1}(z)$ and we may put $h_j(z) = -a_{j1}(z)$.

Now, at least one of the graphs Γ_{h_i} must be contained in $X \cap (V \times \mathbb{C})$, for otherwise $V = \pi(X) \cap V$ would be a finite union of proper analytic subsets of V, which is impossible. Hence $\dim X \geq \dim \Gamma_{h_i} = \dim M$, as required. □

Theorem 5.17. *Suppose X is analytic in $M \times \mathbb{C}^k$, and the projection $\pi|_X : X \to M$ is proper. Then $\dim X = \dim \pi(X)$.*

Proof. By the proof of the Remmert Proper Mapping Theorem, it suffices to consider the case $k = 1$. Let us first show that $\dim \pi(X) \leq \dim X$. Let N be a submanifold of M, contained in $\pi(X)$, and of dimension $\dim N = \dim \pi(X)$. Put $Z = X \cap (N \times \mathbb{C})$. Then the projection $\pi|_Z : Z \to N$ is proper, and $\pi(Z) = N$, so by Lemma 5.16, $\dim Z \geq \dim N = \dim \pi(X)$.

To show that $\dim X \leq \dim \pi(X)$, let S be a submanifold of $M \times \mathbb{C}$, contained in X, and of dimension $\dim S = \dim X$. Since the projection $\pi|_S$ is proper, then the nonempty fibres of $\pi|_S$ are 0-dimensional. By Example 2.5(5), we may assume that the rank of $\pi|_S : S \to M$ is maximal, and hence $\mathrm{rk}(\pi|_S) = \dim S$ (Rank Theorem 1.10). Then

$$\dim X = \dim S = \mathrm{rk}(\pi|_S) = \dim \pi(S) \leq \dim \pi(X)\,.$$

□

6 Local representation of analytic sets

6.1 Normalization and Nullstellensatz

Recall that, for a subset A of a manifold M and a point $\xi \in M$,

$$\dim_\xi A = \min\{\dim(A \cap U) : U \text{ an open neighbourhood of } \xi \text{ in } M\}\,.$$

Theorem 6.1. *Let X be an analytic subset of a manifold M, and let $\xi \in X$. Then*

$$\dim_\xi X \;=\; \min\{\mathrm{codim} N : N \text{ submanifold of } M,\; N \cap X = \{\xi\}\}\,.$$

Proof. Suppose that $k = \min\{\mathrm{codim} N : N \text{ submanifold of } M,\; N \cap X = \{\xi\}\}$, and let N be such that $N \cap X = \{\xi\}$ and $\dim N = m - k$, where $m = \dim M$. The problem being local, without loss of generality we can assume that $M = \epsilon\Delta^k \times \Delta^{m-k}$, $\xi = (0,0)$, $N = \{0\} \times \Delta^{m-k}$, and X is defined in M by some $h_1, \dots, h_s \in \mathcal{O}(M)$ vanishing at $(0,0)$. Let $\pi : \mathbb{C}^k \times \mathbb{C}^{m-k} \to \mathbb{C}^k$ be the canonical projection. Since $X \cap N = \{(0,0)\}$, the fibre $(\pi|_X)^{-1}(0,0)$ is the singleton $\{(0,0)\}$. It follows that, for every $j = k+1, \dots, m$, there exists $1 \le t \le s$ such that the germ $(h_t)_{(0,0)}$ is regular in x_j. Therefore, by Weierstrass Preparation (and after shrinking ϵ if needed), X is a subset of the zero set of a monic polynomial in x_j. Hence, by Lemmas 5.1 and 5.5, for each $j = k+1, \dots, m$, there exists $r_j > 0$ such that

$$x = (x_1, \dots, x_m) \in X \quad \Rightarrow \quad |x_j| < r_j\,.$$

Setting $r = \max\{r_{k+1}, \dots, r_m\}$, we get that

$$x = (x_1, \dots, x_m) \in X \quad \Rightarrow \quad (|x_{k+1}| < r, \dots, |x_m| < r)\,.$$

Thus, by Lemma 5.1 again, $\pi|_X : X \to \mathbb{C}^k$ is proper, and hence $\dim X \le k$, by Theorem 5.17.

On the other hand, if $\dim X < k$, then $\dim \pi(X) < k$, by Theorem 5.17 again, so there exists a complex line $L \subset \mathbb{C}^k$ through 0, for which $\pi(X) \cap L = \{0\}$. Then $\widetilde{N} = N \times L$ is of codimension smaller than k, and $\widetilde{N} \cap X = \{\xi\}$, which contradicts the choice of N. □

Corollary 6.2 (Normalization Lemma)**.** *Let X be an analytic subset of an m-dimensional manifold M. Then, at every point $\xi \in X$, there is a coordinate chart (U, φ) such that $\varphi(\xi) = 0$, $\varphi(U) = \Delta^k \times \Delta^{m-k}$, where $k = \dim_\xi X$, and the projection $\pi|_{\varphi(X)} : \varphi(X) \to \Delta^k$ is proper and surjective.*

Proof. By the above proof, there exists a coordinate chart (U, φ) such that $\pi|_{\varphi(X)} : \varphi(X) \to \Delta^k$ is proper. Then surjectivity of $\pi|_{\varphi(X)}$ follows from the fact that $\pi(\varphi(X))$ is analytic in Δ^k and of dimension k. □

For the next corollary (Ideal Normalization, below), we need to recall the notions of k-normal and k-regular ideals. An ideal I in $\mathcal{O}_n$ is called *k-normal*, when it satisfies the equivalent conditions of the following lemma.

Lemma 6.3. *Let I be an ideal in $\mathcal{O}_n$, and let $0 \le k \le n$. Let $\bar{\mathcal{O}}_k$ denote the image $\mathcal{O}_k/(I \cap \mathcal{O}_k)$ of $\mathcal{O}_k$ under the epimorphism $\mathcal{O}_n \to \mathcal{O}_n/I$. The following conditions are equivalent:*

(i) $\mathcal{O}_n/I$ is finitely generated as a module over $\bar{\mathcal{O}}_k$ (hence also an integral ring extension of $\bar{\mathcal{O}}_k$).

(ii) $\mathcal{O}_n/I \cong \bar{\mathcal{O}}_k[\bar{x}_{k+1}, \dots, \bar{x}_n]$ and the classes (modulo I) $\bar{x}_{k+1}, \dots, \bar{x}_n$ are integral over $\bar{\mathcal{O}}_k$.

(iii) I contains a distinguished polynomial from $\mathcal{O}_k[x_l]$ for every $l = k+1, \dots, n$.

(iv) I contains a regular germ from $\mathcal{O}_l$ for every $l = k+1, \dots, n$.

Proof. **Exercise** (cf. [Ło, III.2]). □

Definition 6.4. An ideal I in $\mathcal{O}_n$ is called *k-regular* when I is k-normal and $\mathcal{O}_k \cap I = 0$.

Corollary 6.5 (Ideal Normalization)**.** *Let $A = \mathcal{V}(I)$ be a k-dimensional analytic germ at $0 \in \mathbb{C}^n$, where $I \triangleleft \mathcal{O}_n$. Then both I and $\mathfrak{J}(A)$ are k-regular, up to an analytic change of coordinates.*

Proof. By Corollary 6.2, we may assume that A has an analytic representative $X \subset \Delta^k \times \Delta^{n-k}$ at 0, such that $\pi|_X : X \to \Delta^k$ is proper and $\pi(X) = \Delta^k$, where $\pi : \Delta^k \times \Delta^{n-k} \to \Delta^k$ is the projection. It follows that $\mathfrak{J}(A) \cap \mathcal{O}_k = 0$, for if $0 \not\equiv f \in \mathfrak{J}(A) \cap \mathcal{O}_k$, then $A = \mathcal{V}(\mathfrak{J}(A)) \subset \mathcal{V}(f)$, and hence $\pi(X) \subset \pi(\{\tilde{f} = 0\})$ would be a proper subset of Δ^k. Hence also $I \cap \mathcal{O}_k = 0$.

Now, as $(\pi|_X)^{-1}(0,0) = \{(0,0)\}$, the ring $\mathbb{C}\{x_{k+1}, \dots, x_n\}/I(0)$ is a finite dimensional $\mathbb{C}$-vector space (where the evaluation is at $x_1 = \dots = x_k = 0$). Then, by Weierstrass Finiteness Theorem 3.17, $\mathcal{O}_n/I$ is a finitely generated $\mathcal{O}_k$-module (hence so is $\mathcal{O}_n/\mathfrak{J}(A)$), and as $I \cap \mathcal{O}_k = 0$ (resp. $\mathfrak{J}(A) \cap \mathcal{O}_k = 0$), it follows that $\mathcal{O}_n/I$ (resp. $\mathcal{O}_n/\mathfrak{J}(A)$) is finitely generated over $\mathcal{O}_k/(I \cap \mathcal{O}_k)$ (resp. $\mathcal{O}_k/(\mathfrak{J}(A) \cap \mathcal{O}_k)$). Thus, by Definition 6.4, I and $\mathfrak{J}(A)$ are k-regular. □

Theorem 6.6 (Nullstellensatz)**.** *For every ideal I in $\mathbb{C}\{x_1, \dots, x_n\}$, we have $\mathfrak{J}(\mathcal{V}(I)) = \mathrm{rad} I$. In particular, $\mathfrak{J}(\mathcal{V}(I)) = I$ when I is prime.*

Proof. Suppose first that I is prime. By Corollary 6.5 above, we may assume that I is k-regular for some $k \geq 0$ and $\mathcal{O}_k \cap \mathfrak{J}(\mathcal{V}(I)) = 0$. Since $I \subset \mathfrak{J}(\mathcal{V}(I))$ anyway, it suffices to show that $\mathfrak{J}(\mathcal{V}(I)) \subset I$. Let $f \in \mathcal{O}_n \setminus I$. Then $\bar{f} \in \mathcal{O}_n/I \setminus \{0\}$, hence by integrality of $\mathcal{O}_n/I$ over $\mathcal{O}_k/(I \cap \mathcal{O}_k)$ (Lemma 6.3), $\bar{f}\bar{g} \in \mathcal{O}_k/(I \cap \mathcal{O}_k) \setminus \{0\}$ for some $g \in \mathcal{O}_n$ (since I is prime, $\mathcal{O}_n/I$ is an integral domain, integral over a *UFD* $\mathcal{O}_k/(I \cap \mathcal{O}_k) = \mathcal{O}_k$, and hence $\bar{f} \in \mathcal{O}_n/I$ has a minimal polynomial $a_0 + a_1 w + \dots + w^s$ over $\mathcal{O}_k/(I \cap \mathcal{O}_k)$; by minimality, $a_0 \neq 0$, so put $g = a_1 + a_2 f + \dots + f^{s-1}$). Therefore $fg \in h + I \subset h + \mathfrak{J}(\mathcal{V}(I))$ for some $h \in \mathcal{O}_k \setminus I$. Now $h \notin \mathfrak{J}(\mathcal{V}(I))$, as $\mathcal{O}_k \cap \mathfrak{J}(\mathcal{V}(I)) = 0$, so $fg \notin \mathfrak{J}(\mathcal{V}(I))$, and thus $f \notin \mathfrak{J}(\mathcal{V}(I))$.

For an arbitrary I, let $I = J_1 \cap \dots \cap J_s$ be a primary decomposition. Then $\mathrm{rad} J_i$ are prime and, by Remarks 4.11.(7)-(8) and 4.14.(2), we get

$$\mathfrak{J}(\mathcal{V}(I)) = \bigcap_i \mathfrak{J}(\mathcal{V}(J_i)) = \bigcap_i \mathfrak{J}(\mathcal{V}(\mathrm{rad} J_i)) = \bigcap_i \mathrm{rad} J_i = \mathrm{rad} I \,.$$

□

6.2 Rückert Lemma

The following result is a consequence of the fact that every finite algebraic field extension L/K can be expressed as a *primitive extension* $L = K[\xi]$.

Proposition 6.7 (Primitive Element)**.** *Suppose A is a UFD and a $\mathbb{C}$-vector space, and $B = A[\eta_1, \dots, \eta_s]$ is integral over A. Then there exists a* primitive element

$$\xi = \sum_{i=1}^{s} a_i \eta_i \in B, \qquad \text{where} \quad a_i \in \mathbb{C}^* \,,$$

such that $\delta B \subset A[\xi]$, where δ is the discriminant of the minimal polynomial of ξ over A.

Proof. See, e.g., [Ło, A.8.3]. □

We are now ready to prove the fundemental result of Rückert.

Proposition 6.8 (Rückert Lemma)**.** *Suppose $A = \mathcal{V}(I)$ is a zero set germ of a prime k-regular ideal I in $\mathcal{O}_n$. Then there exist a connected open neighbourhood U of 0 in $\mathbb{C}^k$, proper analytic subset Z of U, and a representative X of A analytic in $U \times \mathbb{C}^{n-k}$, such that:*

(i) *the projection $\pi|_X : X \to U$ is proper*

(ii) $(\pi|_X)^{-1}(0,0) = \{(0,0)\}$

(iii) *$X \cap \pi^{-1}(U \setminus Z)$ is a k-dimensional manifold, and, locally at every $(x_1, \dots, x_k, x_{k+1}, \dots, x_n) \in X$, a graph of a holomorphic function over a neighbourhood of $(x_1, \dots, x_k)$.*

Proof. For simplicity of notation, let $y = (x_1, \dots, x_k)$ denote the set of variables of $\mathcal{O}_k$. Let $G_1, \dots, G_t \in \mathcal{O}_n$ be a system of generators of I. By Lemma 6.3(iii), I contains polynomials $F_j \in \mathbb{C}\{z\}[x_j]$ distinguished in x_j, for $j = k+1, \dots, n$. Hence, after evaluating at $y = 0$, the diagram $\mathfrak{N}(I(0)) \subset \mathbb{N}^{n-k}$ contains a vertex on each of the axes, and so its complement, $\Delta = \mathbb{N}^{n-k} \setminus \mathfrak{N}(I(0))$, is a finite set. Therefore, by replacing each of the G_i with its remainder after Hironaka Division by the $F_{k+1}, \dots, F_n$, and adding $F_{k+1}, \dots, F_n$ to this collection, we can assume that I is generated by $G_1, \dots, G_s$, where $G_i = G_i(z, x_{k+1}, \dots, x_n) \in \mathbb{C}\{z\}[x_{k+1}, \dots, x_n]$, $i = 1, \dots, s$. Let $\xi \in \mathcal{O}_n$ be such that $\bar{\xi} \in \mathcal{O}_n/I$ is a primitive element of $\mathcal{O}_n/I$ over $\mathcal{O}_k$. By Proposition 6.7 above,

$$\bar{\xi} = \sum_{j=k+1}^{n} a_j \bar{x}_j \qquad \text{for some } a_j \in \mathbb{C}^*\,,$$

so after a linear change of coordinates

$$(x_1, \dots, x_k, x_{k+1}, x_{k+2}, \dots, x_n) \;\mapsto\; (x_1, \dots, x_k, \sum_{j=k+1}^{n} a_j x_j, x_{k+2}, \dots, x_n)\,,$$

I is still k-regular, and $\bar{\xi} = \bar{x}_{k+1}$. Let $F \in \mathcal{O}_k[w]$ be the minimal polynomial of $\bar{x}_{k+1}$, and let $Q_{k+2}, \dots, Q_n \in \mathcal{O}_k[w]$ be the minimal polynomials of $\bar{x}_{k+2}, \dots, \bar{x}_n$ respectively (which exist, as $\mathcal{O}_n/I$ is an integral domain, integral over a *UFD* $\mathcal{O}_k/(I \cap \mathcal{O}_k) = \mathcal{O}_k$). Then $F(x_{k+1}) \in I$ and $Q_j(x_j) \in I$, $j = k+2, \dots, n$. The polynomials $F, Q_{k+2}, \dots, Q_n$ are monic, and hence distinguished. (Indeed, consider Q_n for instance: $Q_n(y, x_n) = x_n^d + c_1(y)x_n^{d-1} + \dots + c_d(y)$ is regular in x_n, and hence, by the Weierstrass Preparation, $Q_n = qQ'_n$, where $q(0,0) \neq 0$ and $Q'_n(y,x) = x_n^l + \gamma_1(y)x_n^{l-1} + \dots + \gamma_l(y)$ is distinguished. Evaluating at $y = 0$, we get that $l = d$, and $c_1(0) = \dots = c_d(0) = 0$.)

Next observe that $F, Q_{k+2}, \dots, Q_n$ are linearly independent modulo $\mathfrak{m}_k I$, since the initial exponents of $F(0)(x_{k+1})$ and $Q_j(0)(x_j)$, $j = k+2, \dots, n$, are pairwise distinct. Therefore we can assume that $F, Q_{k+2}, \dots, Q_n$ are among our generators $G_1, \dots, G_s$ of I (Nakayama Lemma). $F, Q_{k+2}, \dots, Q_n$ being irreducible and distinguished, there are a connected open neighbourhood U of 0 in $\mathbb{C}^k$ and monic irreducible representatives $\widetilde{F}, \widetilde{Q}_{k+2}, \dots, \widetilde{Q}_n \in \mathcal{O}(U)[w]$ of $F, Q_{k+2}, \dots, Q_n$ respectively. By Proposition 5.10, the discriminant $\delta = D(\widetilde{F})$ is not identically zero on U, and hence

$$Z = \{z \in U : \delta(z) = 0\}$$

is a proper analytic subset of U. Moreover, after shrinking U if necessary, we may assume that all the $G_1, \dots, G_s$ have representatives $\widetilde{G}_1, \dots, \widetilde{G}_s \in \mathcal{O}(U \times \mathbb{C}^{n-k})$. Then

$$X = \{\widetilde{G}_1 = \dots = \widetilde{G}_s = 0\}$$

is an analytic representative of A in $U \times \mathbb{C}^{n-k}$, and as

$$X \subset \{\widetilde{F} = \widetilde{Q}_{k+2} = \dots = \widetilde{Q}_n = 0\}\,,$$

the projection $\pi|_X : X \to U$ is proper (Lemma 5.1). Also, $(\pi|_X)^{-1}(0,0) = \{(0,0)\}$, as all the $F, Q_{k+2}, \dots, Q_n$ are distinguished.

By Proposition 6.7 above, $\delta_0 \cdot \mathcal{O}_n/I \subset \mathcal{O}_k[\bar{x}_{k+1}]$. In particular, for $j = k+2, \dots, n$, there are $P_j \in \mathcal{O}_k[w]$ such that

$$\delta_0 x_j - P_j(x_{k+1}) \in I\,.$$

We will now show that, for every $y \in U \setminus Z$,

$$\widetilde{F}(y, x_{k+1}) \;=\; \delta(y)x_{k+2} - \widetilde{P}_{k+2}(y, x_{k+1}) \;=\; \dots \;=\; \delta(y)x_n - \widetilde{P}_n(y, x_{k+1}) \;=\; 0$$
$$\iff\; \widetilde{G}_1(y, x_{k+1}, \dots, x_n) \;=\; \dots \;=\; \widetilde{G}_s(y, x_{k+1}, \dots, x_n) \;=\; 0\,, \quad (6.1)$$

where $\widetilde{P_j} \in \mathcal{O}(U)[w]$ are representatives of the respective P_j. Indeed, the implication "$\Leftarrow$" follows from the fact that the germs at 0 of all the functions of the left hand side belong to $I = (G_1, \dots, G_s)\cdot\mathcal{O}_n$. For the other implication, consider a formal expression $\widetilde{G}_i(y, x_{k+1}, \widetilde{P}_{k+2}(y, x_{k+1})/\delta, \dots, \widetilde{P}_n(y, x_{k+1})/\delta)$. Let $d_i = \deg_x G_i$. Then

$$\delta^{d_i}\widetilde{G}_i(y, x_{k+1}, \widetilde{P}_{k+2}(y, x_{k+1})/\delta, \dots, \widetilde{P}_n(y, x_{k+1})/\delta) \;\in\; \mathcal{O}_k[x_{k+1}]\,,$$

and hence, by minimality of F, we can divide

$$\delta^{d_i}\widetilde{G}_i(y, x_{k+1}, \widetilde{P}_{k+2}(y, x_{k+1})/\delta, \dots, \widetilde{P}_n(y, x_{k+1})/\delta) \;=\; \widetilde{F}(y, x_{k+1})\widetilde{H}_i(y, x_{k+1})$$

for some $H_i \in \mathcal{O}_k[x_{k+1}]$. Therefore, if the left hand side of (6.1) is satisfied, we get

$$\delta^{d_i}\widetilde{G}_i(y, x_{k+1}, \dots, x_n) \;=\; \delta^{d_i}\widetilde{G}_i(y, x_{k+1}, \widetilde{P}_{k+2}(y, x_{k+1})/\delta, \dots, \widetilde{P}_n(y, x_{k+1})/\delta) \;=\; 0\,,$$

and hence $\widetilde{G}_i(y, x_{k+1}, \dots, x_n) = 0$, $i = 1, \dots, s$, as required.

Put now

$$F_1(y, x_{k+1}, \dots, x_n) = \widetilde{F}(y, x_{k+1}) \quad \text{and}$$
$$F_{j-k}(y, x_{k+1}, \dots, x_n) = \delta(y)x_j - \widetilde{P}_j(y, x_{k+1})\,,\; j = k+2, \dots, n\,,$$

and consider $F = (F_1, \dots, F_{n-k}) : (U \setminus Z) \times \mathbb{C}^{n-k} \to \mathbb{C}^{n-k}$. It follows from Lemma 5.9 that, for every $\xi \in X \cap \pi^{-1}(U \setminus Z)$, the Jacobi matrix $\left[\dfrac{\partial F_i}{\partial x_j}(\xi)\right]_{\substack{i=1,\dots,n-k\\ j=k+1,\dots,n}}$ is invertible, and hence, by (6.1), X is a graph of a holomorphic function in a neighbourhood of every such ξ (Implicit Function Theorem). □

7 Irreducibility and dimension

7.1 Thin sets

Definition 7.1. We say that a subset Z of a manifold M is *thin* when Z is closed, nowhere dense, and, for every open $\Omega \subset M$, every function locally bounded on Ω and holomorphic on $\Omega \setminus Z$ extends holomorphically to Ω.

Lemma 7.2. *Assume that $\delta \in \mathcal{O}(M)$ doesn't vanish identically on any component of M, and let Z be the zero set of δ. Then Z is thin in M.*

Proof. Z is closed and nowhere dense, by Theorem 2.7. To prove that Z has the extension property, it suffices to show that, for every $\xi \in Z$, there exists a coordinate neighbourhood U of ξ in M such that every bounded $f \in \mathcal{O}(U \setminus Z)$ admits $F \in \mathcal{O}(U)$ with $F|_{U\setminus Z} \equiv f$. We may thus assume that $M = \mathbb{C}^m$, $\xi = 0$, $f \in \mathcal{O}(R\Delta^m \setminus Z)$ and bounded on $R\Delta^m$ for some $R > 1$.

By Proposition 4.7 and Lemma 5.5, we may assume that the projection $\pi|_Z : Z \to \Delta^{m-1}$ is proper, and $Z \cap (\Delta^{m-1} \times \partial\Delta) = \varnothing$. Let $y = (x_1, \dots, x_{m-1})$, for simplicity of notation. Since f is holomorphic in $\Delta^{m-1} \times \partial\Delta$, we may define

$$F(y, x_m) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(y,\zeta)}{\zeta - x_m} \,\mathrm{d}\zeta \qquad \text{for} \quad (y, x_m) \in \Delta^m .$$

Now F is holomorphic in Δ^m, and we just need to check that $F(y, x_m) = f(y, x_m)$ for $(y, x_m) \in \Delta^m \setminus Z$. Given $y \in \Delta^{m-1}$, the set $Z \cap (\{y\} \times \Delta)$ is finite, so by the Riemann Extension Lemma (see, e.g., [Wh, Lema 3B]), $f(y, \cdot)$ extends to a holomorphic function in $\{y\} \times \Delta$, and is given by the same formula as $F(y, \cdot)$. □

7.2 Analytic cover

Lemma 7.3. *Consider the set $Y = \{v = \eta_1\} \cup \dots \cup \{v = \eta_d\}$ in $(\mathbb{C}^r)^{d+1}$, where $\eta_i = (\eta_i^1, \dots, \eta_i^r)$, $v = (v^1, \dots, v^r) \in \mathbb{C}^r$. There exist polynomials $P_1, \dots, P_s$ symmetric with respect to $\eta_1, \dots, \eta_d$, and such that $Y = \{P_1 = \dots = P_s\}$.*

Proof. By Remark 4.11(7),

$$Y = \{(\eta_1, \dots, \eta_d, v) : f_{(i_1,\dots,i_d)}(\eta_1, \dots, \eta_d, v) = 0 \text{ for all } 1 \le i_1, \dots, i_d \le r\} ,$$

where

$$f_{(i_1,\dots,i_d)}(\eta_1, \dots, \eta_d, v) = (v^{i_1} - \eta_1^{i_1}) \dots (v^{i_d} - \eta_d^{i_d}) .$$

Since Y is symmetric in the η's, it follows that

$$Y = \{(\eta_1, \dots, \eta_d, v) : f_{(i_1,\dots,i_d)}(\eta_{\tau(1)}, \dots, \eta_{\tau(d)}, v) = 0 \text{ for all } 1 \le i_1, \dots, i_d \le r, \tau \in S_d\} .$$

Further, let $l = d!$, and let $\sigma_\nu(w_1, \dots, w_l) = (-1)^\nu \sum\limits_{j_1 < \dots < j_\nu} w_{j_1} \dots w_{j_\nu}$, $\nu = 1, \dots, l$, be the Viéte's polynomials in l variables. Put $\sigma = (\sigma_1, \dots, \sigma_l) : \mathbb{C}^l \to \mathbb{C}^l$. Then, since $\sigma^{-1}(0) = 0$, we have

$$Y = \{(\eta_1, \dots, \eta_d, v) : P_{\nu, i_1, \dots, i_d} = 0 \text{ for all } 1 \le i_1, \dots, i_d \le r,\ \nu = 1, \dots, l\} ,$$

where $P_{\nu,i_1,\dots,i_d} := \sigma_\nu(f_{(i_1,\dots,i_d)}(\eta_{\tau_1(1)}, \dots, \eta_{\tau_1(d)}, v), \dots, f_{(i_1,\dots,i_d)}(\eta_{\tau_l(1)}, \dots, \eta_{\tau_l(d)}, v))$.

Now $P_{\nu,i_1,\dots,i_d}$ are clearly all symmetric with respect to $\eta_1,\dots,\eta_d$, and, for $x=(x_1,\dots,x_r)\in\mathbb{C}^r$, we have

$$P_{\nu,i_1,\dots,i_d}(\eta_1,\dots,\eta_d,x)=\sum_{\substack{\beta\in\mathbb{N}^r\\|\beta|\le\nu d}} a_\beta(\eta_1,\dots,\eta_d)x^\beta\,,$$

where a_β are also symmetric with respect to $\eta_1,\dots,\eta_d$. □

Lemma 7.4 (Analytic Cover)**.** *Suppose M is a connected manifold, Z is a thin subset of M, and N is a closed submanifold of $(M\setminus Z)\times\mathbb{C}^r$, which is locally a graph of a holomorphic function over $M\setminus Z$, and such that the projection $\pi|_{\overline{N}}:\overline{N}\to M$ is proper. Then, for every collection $\{\Lambda_s\}_{s\in S}$ of connected components of N, the set $\overline{\bigcup_{s\in S}\Lambda_s}$ is analytic in $M\times\mathbb{C}^r$.*

Proof. Observe first that, for every $S\neq\varnothing$, $\bigcup_{s\in S}\Lambda_s$ itself is a closed submanifold of $(M\setminus Z)\times\mathbb{C}^r$, which is locally a graph of a holomorphic function over $M\setminus Z$, and that the projection $\pi|_{\overline{\bigcup_s\Lambda_s}}:\overline{\bigcup_{s\in S}\Lambda_s}\to M$ is proper if $\pi|_{\overline{N}}:\overline{N}\to M$ is so. Therefore it suffices to prove the result for N itself.

The projection $\pi|_N:N\to M\setminus Z$ being proper, it is in fact a finite locally biholomorphic cover. It follows from Definition 7.1 that $M\setminus Z$ is connected, and hence the rank of the cover is constant over $M\setminus Z$; say, equal d. Let now $P_1(\eta_1,\dots,\eta_d,v),\dots,P_s(\eta_1,\dots,\eta_d,v)$ be the symmetric polynomials of Lemma 7.3. For $(y,x)\in(M\setminus Z)\times\mathbb{C}^r$, define

$$F_j(y,x)=P_j(\eta_1(y),\dots,\eta_d(y),x)\,,\quad j=1,\dots,s\,,$$

where $\{\eta_1(y),\dots,\eta_d(y)\}=N\cap(\pi|_N)^{-1}(y)$. Then by Lemma 7.3, N is precisely the set of common zeros of $F_1,\dots,F_s$ in $(M\setminus Z)\times\mathbb{C}^r$. Note that the F_j are holomorphic as composites of holomorphic functions ($\eta_1,\dots,\eta_d$ being the holomorphic functions locally defining N over $M\setminus Z$).

Moreover, $F=(F_1,\dots,F_s)$ extends (uniquely) to a holomorphic mapping $M\times\mathbb{C}^r\to\mathbb{C}^s$. Indeed, for $j=1,\dots,s$, $P_j(\eta_1,\dots,\eta_d,x)=\sum a_\beta(\eta_1,\dots,\eta_d)x^\beta$, where the sum is finite and the a_β are symmetric polynomials in $\eta_1,\dots,\eta_d$. Now $\eta_1(y),\dots,\eta_d(y)$ being holomorphic functions on $M\setminus Z$, it follows that $a_\beta(\eta_1(y),\dots,\eta_d(y))\in\mathcal{O}(M\setminus Z)$. By properness of $\pi|_{\overline{N}}$, the a_β are also locally bounded on M, and hence extend uniquely to $\mathcal{O}(M)$, by Definition 7.1.

We will now show that $\overline{N}=F^{-1}(0)$, which will complete the proof. Consider the coordinate projection $\tilde{\pi}:F^{-1}(0)\to M$. We have $\tilde{\pi}^{-1}(M\setminus Z)=N$, and hence it suffices to show that $\tilde{\pi}^{-1}(M\setminus Z)$ is dense in $F^{-1}(0)$. Suppose otherwise. Then there exist $(y_0,x_0)\in F^{-1}(0)$, and its relatively compact open neighbourhood $U\times V\subset M\times\mathbb{C}^r$, such that $(\overline{U}\times\overline{V})\cap\tilde{\pi}^{-1}(M\setminus Z)=\varnothing$. We may then choose $(y_n)_1^\infty\subset U\setminus Z$ such that $y_n\xrightarrow[n\to\infty]{}y_0$ and $F(y_n,x)\neq0$ for any $x\in\overline{V}$. (Indeed, if $y_n\in U\setminus Z$, then $F(y_n,x)=0$ iff $(y_n,x)\in N=\tilde{\pi}^{-1}(M\setminus Z)$; but $(y_n,x)\in\overline{U}\times\overline{V}$, so $(y_n,x)\notin N$.)

Hence, for every $n\in\mathbb{N}$ and every $x\in\overline{V}$, there is $1\le j\le s$ with

$$P_j(\eta_1(y_n),\dots,\eta_d(y_n),x)=F_j(y_n,x)\neq0\,,$$

and thus $\eta_i(y_n)\notin\overline{V}$, $i=1,\dots,d$. By properness of $\pi|_{\overline{N}}$ again, for every $i=1,\dots,d$, the sequence $(\eta_i(y_n))_{n=1}^\infty$ is bounded, and so we can, for every $i=1,\dots,d$, choose a convergent subsequence $\eta_i(y_n)\xrightarrow[n\to\infty]{}\eta_i\neq x_0$. Hence

$$F(y_0,x_0)=(P_1(\eta_1,\dots,\eta_d,x_0),\dots,P_s(\eta_1,\dots,\eta_d,x_0))\neq0\,,$$

which contradicts the choice of (y_0,x_0). □

7.3 Local irreducibility and dimension

Remark 7.5. Clearly, every connected locally irreducible analytic set is irreducible. The converse, however, is usually not true. Consider, for instance, the following algebraic curve in $\mathbb{C}^2$,

$$X = \{(x,y) \in \mathbb{C}^2 : y^2 = x^2(x-1)\}\,.$$

Then X is irreducible (**Exercise**), but $X_{(0,0)}$ splits into two irreducible components $y = \pm x\sqrt{x-1}$, where $\sqrt{x-1}$ denotes a fixed branch of the square root of $x-1$ at zero.

Proposition 7.6. *Suppose that an analytic subset X of a manifold M, and a point $\xi \in X$ are such that X_ξ is irreducible, of dimension k. Then there exists a neighbourhood Ω of ξ in M for which* $\mathrm{reg}(X \cap \Omega)$ *is a connected k-submanifold of Ω, and* $\dim Y < k$ *for every proper analytic subset Y of $X \cap \Omega$.*

Proof. The problem being local, we may assume that $M = \Delta^m$ and $\xi = 0$. Then, by Corollaries 6.2 and 6.5, and Rückert Lemma 6.8, there are local coordinates $(x_1, \dots, x_m)$ at 0, a connected open neighbourhood U of 0 in $\mathbb{C}^k$, and a proper analytic subset Z of U such that the restriction $\pi|_X : X \to U$ of the coordinate projection $\pi : \Delta^k \times \Delta^{m-k} \to \Delta^k$ is proper, $(\pi|_X)^{-1}(0) = 0$, and $X \cap \pi^{-1}(U \setminus Z)$ is a k-dimensional manifold, and locally a graph of a holomorphic function over $U \setminus Z$. Call this manifold N, and let $\Omega = U \times \Delta^{m-k}$.

By irreducibility of X_ξ and Lemma 7.4, we get that N is connected, and the proper analytic subset $Z' = X \cap \Omega \cap \pi^{-1}(Z)$ of X is contained in $\overline{N}$. Moreover, as a connected submanifold of X, N is contained in a connected component, say Λ_1, of $\mathrm{reg}(X \cap \Omega)$. Suppose $\mathrm{reg}(X \cap \Omega)$ has another component Λ_2. Then Λ_2 must be disjoint from N, and hence contained in Z'. But then $\Lambda_2 \subset Z' \subset \overline{N} \subset \overline{\Lambda}_1$, contradicting the openness of Λ_2 in $X \cap \Omega$. Thus $\mathrm{reg}(X \cap \Omega)$ is connected.

Let now Y be a proper analytic subset of $X \cap \Omega$. Suppose that $\dim Y = k$. Then $Y \not\subset Z'$, because the properness of projection $\pi|_{Z'} : Z' \to Z$ implies that $\dim Z' = \dim Z < k$ (Theorem 5.17). Hence $Y \cap \mathrm{reg}(X \cap \Omega)$ is an analytic subset of a connected manifold $\mathrm{reg}(X \cap \Omega)$, with nonempty interior, and thus $Y \cap \mathrm{reg}(X \cap \Omega) = \mathrm{reg}(X \cap \Omega)$. Therefore $Y \supset \overline{Y \cap \mathrm{reg}(X \cap \Omega)} = X$; a contradiction. □

Corollary 7.7. *Suppose that an analytic subset X of a manifold M and a point $\xi \in X$ are such that X_ξ is irreducible, of dimension k. Then there exists a neighbourhood Ω of ξ in M for which*

$$\dim_x \mathrm{sng}(X \cap \Omega) < \dim_x(X \cap \Omega) \quad \text{for all } x \in \Omega\,,$$

and $X \cap \Omega$ is of pure dimension k.

Proof. Let Ω and Z' be as in the proof of Proposition 7.6 above. Then $\mathrm{sng}(X \cap \Omega) \subset Z'$ and every point of $X \cap \Omega$ lies in the closure of the k-dimensional manifold $\mathrm{reg}(X \cap \Omega)$, hence the result. □

Theorem 7.8 (Irreducible Components)**.** *Let X be an analytic subset of a manifold M. Then the family $\{\Lambda_s\}_{s \in S}$ of connected components of* $\mathrm{reg}X$ *is locally finite in M, and, for every $T \subset S$, $\overline{\bigcup_{s \in T} \Lambda_s}$ is analytic in M. The sets $\bar{\Lambda}_s$, $s \in S$, called the* irreducible components *of X, are irreducible, and $X = \bigcup_{s \in S} \bar{\Lambda}_s$.*

Proof. Let $\xi \in X$, and let $X_\xi = (X_1)_\xi \cup \dots \cup (X_q)_\xi$ be the decomposition into irreducible components (Proposition 4.18). We can now choose a neighbourhood Ω of ξ in M, and representatives $X_1, \dots, X_q$ of $(X_1)_\xi, \dots, (X_q)_\xi$ respectively, analytic in Ω and such that

- $X \cap \Omega = X_1 \cup \cdots \cup X_q$,
- $\mathrm{reg} X_j$ is a connected submanifold of Ω, and
- $\dim Y < \dim X_j$ for every proper analytic subset Y of X_j,

for all $j = 1, \dots, q$, by Proposition 7.6 above.

Fix $i \in \{1, \dots, q\}$. By Proposition 4.18, $X_i \cap \bigcup_{j \neq i} X_j$ is a proper analytic subset of X_i, and hence $\dim(X_i \cap \bigcup_{j \neq i} X_j) < \dim X_i$. Then $(\mathrm{reg} X_i) \cap \bigcup_{j \neq i} X_j$ is a proper analytic subset of the manifold $\mathrm{reg} X_i$, so by Theorem 2.7, $\Theta_i = (\mathrm{reg} X_i) \setminus \bigcup_{j \neq i} X_j$ is again a connected submanifold of Ω, and $\overline{\Theta}_i = X_i$ is analytic in Ω. Clearly,

$$\begin{aligned} \mathrm{reg} X \cap \Omega &= \mathrm{reg}(X \cap \Omega) = \mathrm{reg}(X_1 \cup \cdots \cup X_q) \\ &= \left(\bigcup_{i=1}^{q} \mathrm{reg} X_i \right) \setminus \left(\bigcup_{i=1}^{q} \bigcup_{j \neq i} X_i \cap X_j \right) = \bigcup_{i=1}^{q} \left((\mathrm{reg} X_i) \setminus \bigcup_{j \neq i} X_j \right) , \end{aligned} \tag{7.1}$$

hence $\Theta_1, \dots, \Theta_q$ are precisely the connected components of $\mathrm{reg}(X \cap \Omega)$.

Now, consider the family $\{\Lambda_s\}_{s \in S}$ of connected components of $\mathrm{reg} X$. The set X being closed in M, the family is locally finite in M iff it is locally finite in X. The local finiteness now follows from the fact that, for every $\xi \in X$ and its neighbourhood Ω^ξ as above, we have, for every $s \in S$, either $\Lambda_s \cap \Omega^\xi = \varnothing$ or else $\Lambda_s \cap \Omega^\xi = \Theta_{i_1} \cup \cdots \cup \Theta_{i_s}$ for some $i_1, \dots, i_s \in \{1, \dots, q_\xi\}$ (by (7.1)). The closures $\overline{\Lambda}_s$ are thus analytic in M. Each $\overline{\Lambda}_s$ is also irreducible, because if $\overline{\Lambda}_s = Y_1 \cup Y_2$ then one of those summands, say Y_1, must contain an open subset of $\overline{\Lambda}_s$ and hence of Λ_s; but then $Y_1 \cap \Lambda_s = \Lambda_s$ (by Theorem 2.7), so $Y_1 \supset \overline{Y_1 \cap \Lambda_s} = \overline{\Lambda}_s$. Finally,

$$X = \bigcup_{\xi \in X} X \cap \Omega^\xi = \bigcup_{\xi \in X} \bigcup_{i=1}^{q_\xi} \overline{\Theta}_i = \bigcup_{s \in S} \overline{\Lambda}_s .$$

□

We can now "globalize" previous results on germs to analytic sets:

Corollary 7.9. *An analytic subset X of a manifold M is irreducible iff* $\mathrm{reg} X$ *is connected.*

Corollary 7.10. *If X is an irreducible analytic subset of a manifold M, then X is of pure dimension,* $\dim \mathrm{sng} X < \dim X$*, and* $\dim Y < \dim X$ *for every proper analytic subset Y of X.*

Proof. By Corollary 7.9, $\mathrm{reg} X$ is a connected submanifold of M; say, of dimension k. Therefore, to prove that X is of pure dimension k and $\dim \mathrm{sng} X < \dim X$, it suffices to show that, for every $\xi \in X$, $\dim_\xi \mathrm{sng} X < \dim_\xi X$. Let then ξ be a point of X and let Ω be a neighbourhood of ξ in M, as in the proof of Theorem 7.8. Then, by (7.1), we have

$$\mathrm{sng} X \cap \Omega = \mathrm{sng}(X \cap \Omega) = \bigcup_{i=1}^{q} \left(\mathrm{sng} X_i \cup \bigcup_{j \neq i} (X_i \cap X_j) \right) , \tag{7.2}$$

where $X_1, \dots, X_q$ are the representatives in Ω of the irreducible components of X_ξ. Now, for each $i = 1, \dots, q$, $\dim_\xi \mathrm{sng} X_i < \dim_\xi X_i$, by Corollary 7.7, and $\dim_\xi (X_i \cap X_j) < \dim_\xi X_i$, by Proposition 7.6. Hence

$$\dim_\xi \mathrm{sng} X < \max_{i=1,\dots,q} \dim_\xi X_i \leq \dim_\xi X \,.$$

Let now Y be an analytic subset of X, and suppose that $\dim Y = \dim X$. Then $Y \not\subset \mathrm{sng} X$, by above, so $Y \cap \mathrm{reg} X$ is a nonempty analytic subset of a connected k-manifold. But $\dim Y = k$ implies that Y contains an open subset of $\mathrm{reg} X$, and hence, by Theorem 2.7, $Y \cap \mathrm{reg} X = \mathrm{reg} X$. Then $Y = \overline{\mathrm{reg} X} = X$, which completes the proof. □

Theorem 7.11. *For every family $\{X_s\}_{s \in S}$ of analytic subsets of a manifold M, the intersection $\bigcap_{s \in S} X_s$ is analytic in M.*

Proof. It suffices to show that, for every $\xi \in M$, there exist a coordinate neighbourhood U and a finite subset $T \subset S$ such that

$$U \cap \bigcap_{s \in S} X_s \;=\; U \cap \bigcap_{s \in T} X_s \,.$$

Given $\xi \in M$, let then U be its relatively compact coordinate neighbourhood. Then, for every finite $T \subset S$, the analytic set $U \cap \bigcap_{s \in T} X_s$ has finitely many irreducible components in U, by local finiteness in Theorem 7.8. For every such T, define

$$\nu(T) \;=\; (\nu_m(T), \dots, \nu_0(T)) \in \mathbb{N}^{m+1} \,, \qquad \text{where} \;\; m = \dim M \,,$$

and $\nu_k(T)$ is the number of irreducible components of $U \cap \bigcap_{s \in T} X_s$ of dimension k. Observe that, for finite subsets $T_1, T_2 \subset S$, we have

$$T_1 \subset T_2 \implies \nu(T_1) \geq_{\mathrm{lex}} \nu(T_2) \,.$$

Indeed, if h is one of the defining functions of X_t, where $t \in T_2 \setminus T_1$, then either h vanishes identically on all the components of $U \cap \bigcap_{s \in T_1} X_s$, or else its zero set along a component Y_k (of dimension k, say) is a proper subset of Y_k, hence $\nu_k(T_2) < \nu_k(T_1)$. Therefore there exists a finite set $T_0 \subset S$ for which

$$\nu(T_0) = \min{}_{\mathrm{lex}} \{\nu(T) : \; T \subset S, \; \#T < \infty\} \,,$$

and hence $U \cap \bigcap_{s \in S} X_s \;=\; U \cap \bigcap_{s \in T_0} X_s$. □

8 Coherent sheaves

8.1 Presheaves and sheaves

Unless otherwise specified, all rings are assumed commutative with unity, and the ring homomorphisms are unity preserving.

Definition 8.1. Let X be a topological space. A *presheaf of rings* $\mathcal{F}$ (*on* X) is a family $\{\mathcal{F}(U)\}_{U\in\mathfrak{Top}X}$ of rings with the following properties:

(1) $\mathcal{F}(\varnothing) = 0$

(2) For every pair of open sets $V \subset U$, there is a ring homomorphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that

(a) $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$, and

(b) if $W \subset V \subset U$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

The elements of $\mathcal{F}(U)$ are called *sections* of $\mathcal{F}$ on U, and ρ_{UV} are called *restrictions.* We will write $s|_V$ instead of $\rho_{UV}(s)$, for short.

Definition 8.2. A *sheaf* on a topological space X is a presheaf $\mathcal{F}$ satisfying in addition:

(3) For every open cover $\{U_i\}_{i\in I}$ of an open set $U \subset X$, and a section $s \in \mathcal{F}(U)$,

$$s|_{U_i} = 0 \text{ for all } i \in I \quad \Rightarrow \quad s = 0$$

(4) For every open cover $\{U_i\}_{i\in I}$ of an open set $U \subset X$, and sections $s_i \in \mathcal{F}(U_i)$, if

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \text{ for all } i, j \in I,$$

then there is a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Definition 8.3. Let $\mathcal{F}$ be a presheaf on a topological space X, and let $\xi \in X$. A *stalk* of $\mathcal{F}$ at ξ is

$$\mathcal{F}_\xi = \varinjlim_{\xi\in U} \mathcal{F}(U)\,,$$

where the direct limit is taken over all open neighbourhoods of ξ. The elements f_ξ of $\mathcal{F}_\xi$ are called the *germs* of $\mathcal{F}$ at ξ. Equivalently, $\mathcal{F}_\xi$ can be thought of as the set

$$\{(U, f) : U \text{ open neighbourhood of } \xi, f \in \mathcal{F}(U)\}$$

modulo the equivalence relation

$$(U, f) \sim (V, g) \quad \Leftrightarrow \quad \text{there is an open neighbourhood } W \subset U \cap V \text{ of } \xi, \text{ such that } f|_W = g|_W\,.$$

Remark 8.4. In the same manner one defines (pre)sheaves of sets, abelian groups, or objects of any fixed category $\mathfrak{C}$. The stalks are then sets, abelian groups, or objects of $\mathfrak{C}$ respectively.

Example 8.5.

1. The family $\{\mathcal{O}(U) : U \text{ open in } \mathbb{C}^n\}$ of rings of holomorphic functions forms a sheaf (with natural restrictions), denoted $\mathcal{O}$. The stalks $\mathcal{O}_a$ of $\mathcal{O}$ are isomorphic with $\mathbb{C}\{x-a\}$ (via Taylor expansion at a).

2. Similarly, given any topological space X, the family $\{f : U \to \mathbb{C} \,|\, U \text{ open in } X, f \in \mathcal{C}(U)\}$ of rings of continuous complex valued functions forms a sheaf, denoted $\mathcal{C}$.

3. The family $\{f : U \to \mathbb{Z} \,|\, f \text{ continuous}, U \text{ open in } X\}$, where X is a topological space, forms a sheaf called the *sheaf of (locally) constant functions*, denoted by $\underline{\mathbb{Z}}$. For every $x \in X$, $\underline{\mathbb{Z}}_x \cong \mathbb{Z}$. If α is the number of connected components of X, then $\underline{\mathbb{Z}}(X) \cong \mathbb{Z}^\alpha$.
   In general, given a ring A, equipped with the discrete topology, we define the *constant sheaf* $\mathcal{A}$ by
   $$\mathcal{A}(U) = \{f : U \to A \,|\, f \text{ continuous}\} \qquad \text{for all open } U \subset X\,.$$
   NB. The family $\{f : U \to \mathbb{Z} \,|\, f \text{ constant}\}_{U \in \mathfrak{Top}X}$ forms a presheaf but not a sheaf if X is not connected.

4. Let A be a ring, X a topological space, and $\xi \in X$. A *skyscraper sheaf* at ξ is defined as
   $$A_\xi(U) = \begin{cases} A & : \xi \in U \\ 0 & : \xi \notin U\,. \end{cases}$$
   We have $(A_\xi)_x = A$ iff $x \in \overline{\{\xi\}}$, and $(A_\xi)_x = 0$ otherwise.

Definition 8.6. A *morphism* of presheaves $\alpha : \mathcal{F} \to \mathcal{G}$ on X is a family $\{\alpha(U) : \mathcal{F}(U) \to \mathcal{G}(U)\}_{U \in \mathfrak{Top}X}$ of ring homomorphisms, such that, for every pair of open $V \subset U$, the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\alpha(U)} & \mathcal{G}(U) \\ {\scriptstyle\rho_{UV}}\big\downarrow & & \big\downarrow{\scriptstyle\rho'_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\alpha(V)} & \mathcal{G}(V)\,. \end{array}$$

A morphism α is said to be an *isomorphism* when it has a two-sided inverse.

By the commutativity of the above diagram, $\alpha : \mathcal{F} \to \mathcal{G}$ induces, for every $x \in X$, a homomorphism of stalks $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$. The local nature of sheaves manifests in the following simple yet fundamental property:

Proposition 8.7. *A morphism of sheaves $\alpha : \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ is isomorphic for every $x \in X$.*

Proof. The "only if" being clear, suppose that $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ is isomorphic for all $x \in X$. To show that α is an isomorphism, it suffices to show that $\alpha(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is isomorphic for every open $U \subset X$, because then we can define the inverse morphism β by $\beta(U) = \alpha(U)^{-1}$ for all U.

We will first show that $\alpha(U)$ is injective. Choose $s \in \mathcal{F}(U)$ for which $\alpha(U).s = 0$. Then, for every $x \in U$,
$$\alpha_x(s_x) = (\alpha(U).s)_x = 0\,,$$
hence $s_x = 0$ for $x \in U$, by injectivity of α_x. This is to say that, for every $x \in U$, there is a neighbourhood $U^x \subset U$ such that $s|_{U^x} = 0|_{U^x} = 0$, and thus $s = 0$, by property (3) of sheaves.

Now, for the surjectivity of $\alpha(U)$, let $t \in \mathcal{G}(U)$. For every $x \in U$, there exists $\bar{s}^x \in \mathcal{F}_x$ for which $\alpha_x(\bar{s}^x) = t_x$, by surjectivity of α_x. Choose representatives (U^x, s^x) of $\bar{s}^x$ (i.e., $U^x \subset U$ open, and $(s^x)_x = \bar{s}^x$). Since

$$(\alpha(U^x).s^x)_x = \alpha_x(\bar{s}^x) = t_x ,$$

then after shrinking U^x if necessary, we may assume that $\alpha(U^x).s^x = t|_{U^x}$. Then $s^x|_{U^x \cap U^y} = s^y|_{U^x \cap U^y}$ for $x, y \in U$, by injectivity of α proven above, because both the restrictions are sent to $t|_{U^x \cap U^y}$. Now, by property (4) of sheaves, there is a section $s \in \mathcal{F}(U)$ such that $s|_{U^x} = s^x$ for $x \in U$. We claim that $\alpha(U).s = t$. Indeed, for every $x \in U$, $(\alpha(U).s)|_{U^x} = \alpha(U^x).s^x = t|_{U^x}$, and hence the two sections agree on U, by the sheaf property (3) again. □

Proposition 8.8. *Given a presheaf $\mathcal{F}$, there exist a unique sheaf $\mathcal{F}^+$ and a morphism of presheaves $\vartheta : \mathcal{F} \to \mathcal{F}^+$, such that, for every sheaf $\mathcal{G}$ and a morphism $\gamma : \mathcal{F} \to \mathcal{G}$ of presheaves, there is a unique β satisfying $\beta \circ \vartheta = \gamma$.*

Proof. Uniqueness is a standard **Exercise**. For the proof of existence, one checks that the following satisfies the conditions of the proposition:

$$\begin{aligned}\mathcal{F}^+(U) = \{s : U \to \coprod_{x\in U} \mathcal{F}_x \mid\ & s(x) \in \mathcal{F}_x \text{ for all } x \in U, \text{ and} \\ & \text{for } x \in U \text{ there are a nbhd } V \subset U \text{ of } x \text{ and } t \in \mathcal{F}(V) \text{ st. } s(y) = t_y \text{ for } y \in V\}\end{aligned}$$

for all nonempty open $U \subset X$, and $\mathcal{F}^+(\varnothing) = 0$. □

Remark 8.9. $\mathcal{F}^+$ above is called the *sheafification* of $\mathcal{F}$. It follows directly from the construction that $\mathcal{F}^+_\xi = \mathcal{F}_\xi$ for all $\xi \in X$.

Example 8.10. The "constant sheaf" of Example 8.5.3 is the sheafification of

$$\{f : U \to A \mid f \equiv \text{const}\}_{U \in \mathfrak{Top}X} .$$

Definition 8.11. A *subsheaf* of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ such that, for every open $U \subset X$, $\mathcal{F}'(U)$ is a subring of $\mathcal{F}(U)$, and the restriction maps of $\mathcal{F}'$ are induced by those of $\mathcal{F}$. It follows that, for any point $\xi \in X$, the stalk $\mathcal{F}'_\xi$ is a subring of $\mathcal{F}_\xi$.

Definition 8.12. Given a morphism of sheaves $\alpha : \mathcal{F} \to \mathcal{G}$ on X, we define the *kernel*, *cokernel*, and *image of* α as the sheafification of the presheaves $\{\ker \alpha(U)\}_{U \in \mathfrak{Top}X}$, $\{\operatorname{coker} \alpha(U)\}_{U \in \mathfrak{Top}X}$, $\{\operatorname{im} \alpha(U)\}_{U \in \mathfrak{Top}X}$ respectively.

Remark 8.13.

(1) The presheaf $\ker \alpha$ is, in fact, a sheaf (**Exercise**).

(2) Due to the unique factorization in the definition of sheafification, we can regard the sheaf $\operatorname{im} \alpha$ as a subsheaf of $\mathcal{G}$.

Definition 8.14.
- A morphism of sheaves $\alpha : \mathcal{F} \to \mathcal{G}$ is called *injective* when $\ker \alpha = 0$. Equivalently, by Remark 8.13(1), α is injective iff $\alpha(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is so for every open $U \subset X$.
- A morphism $\alpha : \mathcal{F} \to \mathcal{G}$ is called *surjective* when (after the identification of Remark 8.13(2)) $\operatorname{im} \alpha = \mathcal{G}$.

- We say that a sequence
$$\ldots \longrightarrow \mathcal{F}^{i-1} \stackrel{\alpha^{i-1}}{\longrightarrow} \mathcal{F}^{i} \stackrel{\alpha^{i}}{\longrightarrow} \mathcal{F}^{i+1} \longrightarrow \ldots$$
is *exact at* $\mathcal{F}^i$ when $\operatorname{im}\alpha^{i-1} = \ker \alpha^i$.

- If $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$, we define the *quotient sheaf* $\mathcal{F}/\mathcal{F}'$ to be the sheafification of the presheaf $\{U \mapsto \mathcal{F}(U)/\mathcal{F}'(U)\}_{U\in \mathfrak{Top}X}$.

8.2 $\mathbb{C}$-ringed spaces

Definition 8.15. A *ringed space* is a pair $X = (|X|, \mathcal{O}_X)$ consisting of a topological space $|X|$ and a sheaf of rings $\mathcal{O}_X$ on $|X|$, called the *structure sheaf* of X. It is called a *locally ringed space* when, for every $\xi \in |X|$, the stalk $\mathcal{O}_{X,\xi}$ is a local ring. Its maximal ideal is denoted by $\mathfrak{m}_{X,\xi}$. A locally ringed space is called a $\mathbb{C}$*-ringed space* when furthermore $\mathcal{O}_X$ is a sheaf of $\mathbb{C}$-algebras and, for every $\xi \in |X|$, there is an isomorphism
$$\mathcal{O}_{X,\xi}/\mathfrak{m}_{X,\xi} \cong \mathbb{C}$$
of $\mathbb{C}$-algebras.

Remark 8.16.

(1) For simplicity of notation, we will often write X instead of $|X|$ for the underlying topological space of $(|X|, \mathcal{O}_X)$.

(2) If $U \subset X$ is an open subset, then U together with the restriction $\mathcal{O}_U = \mathcal{O}_X|_U$ is again a ringed space.

Definition 8.17. If X is a ringed space, any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called an $\mathcal{O}_X$*-module* or a *sheaf of modules over* X. For $U \subset X$ open, $\mathcal{F}(U)$ denotes the $\mathcal{O}_X(U)$-module of sections of $\mathcal{F}$ over U. If $f \in \mathcal{O}_X(U)$ and $\mathcal{O}_{X,\xi}$ is local,
$$f(\xi) = f_\xi + \mathfrak{m}_{X,\xi} \ \in \ \mathcal{O}_{X,\xi}/\mathfrak{m}_{X,\xi}$$
is called the *value* of f in ξ.

Definition 8.18. Given ringed spaces X and Y, a continuous mapping $\varphi : |X| \to |Y|$, and an $\mathcal{O}_X$-module $\mathcal{F}$, the presheaf
$$\mathfrak{Top}Y \ni V \ \mapsto \ \mathcal{F}(\varphi^{-1}(V))$$
on Y is a sheaf. We denote it by $\varphi_*\mathcal{F}$ and call it the *direct image* of $\mathcal{F}$. It is a $\varphi_*\mathcal{O}_X$-module.
If $\alpha : \mathcal{F} \to \mathcal{G}$ is a homomorphism of $\mathcal{O}_X$-modules, we define
$$\varphi_*\alpha : \varphi_*\mathcal{F} \to \varphi_*\mathcal{G}$$
by $(\varphi_*\alpha)(V) = \alpha(\varphi^{-1}(V))$. Then $\varphi_*\alpha$ is a homomorphism of $\varphi_*\mathcal{O}_X$-modules.

If $\mathcal{G}$ is an $\mathcal{O}_Y$-module, we define the *(topological) inverse image* of $\mathcal{G}$, denoted $\varphi^{-1}(\mathcal{G})$, as the sheafification of the presheaf
$$\mathfrak{Top}X \ni U \ \mapsto \ \varinjlim_{\substack{V\,\text{open} \\ V\supset\varphi(U)}} \mathcal{G}(V)\,,$$
which is thus uniquely determined by the property $(\varphi^{-1}\mathcal{G})_\xi = \mathcal{G}_{\varphi(\xi)}$ for all $\xi \in X$. Clearly, $\varphi^{-1}(\mathcal{G})$ is an $\varphi^{-1}(\mathcal{O}_Y)$-module.

Definition 8.19. A *morphism* $\varphi : X \to Y$ *of ringed spaces* $X = (|X|, \mathcal{O}_X)$ and $Y = (|Y|, \mathcal{O}_Y)$ is a pair $\varphi = (|\varphi|, \varphi^*)$ consisting of a continuous map

$$|\varphi| : |X| \to |Y|$$

and a homomorphism

$$\varphi^* : \mathcal{O}_Y \to |\varphi|_* \mathcal{O}_X$$

of sheaves of rings on Y.

We think of φ^*_ξ (for $\xi \in X$) as the ring homomorphism

$$\varphi^*_\xi : \mathcal{O}_{Y,\varphi(\xi)} \to \mathcal{O}_{X,\xi}\,,$$

defined as the composition of the canonical homomorphisms

$$\mathcal{O}_{Y,\varphi(\xi)} \;\to\; (|\varphi|_* \mathcal{O}_X)_{\varphi(\xi)} \;\to\; \mathcal{O}_{X,\xi}\,.$$

In case X and Y are locally ringed spaces, a morphism by definition has to be *local*, that is, satisfy

$$\varphi^*_\xi(\mathfrak{m}_{Y,\varphi(\xi)}) \subset \mathfrak{m}_{X,\xi}$$

for every $\xi \in X$.

A *morphism of* $\mathbb{C}$*-ringed spaces* X and Y is a morphism of ringed spaces, where φ^* is furthermore a homomorphism of sheaves of $\mathbb{C}$-algebras. In this case φ^*_ξ is automatically local for every $\xi \in X$.

We obtain the categories of ringed spaces, locally ringed spaces, and $\mathbb{C}$-ringed spaces.

Lemma 8.20. *Let* $\varphi : X \to Y$ *be a morphism of* $\mathbb{C}$*-ringed spaces. Then* φ *is an isomorphism if and only if* $|\varphi|$ *is a homeomorphism and* φ^*_ξ *is an isomorphism for every* $\xi \in X$*.*

Proof. **Exercise**. □

8.3 Basic properties of coherent sheaves

Definition 8.21. Let $X = (|X|, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ a sheaf of modules over X. $\mathcal{F}$ is called of *finite type* (resp. *locally free*) when, for every $\xi \in X$, there is an open neighbourhood U and an epimorphism (resp. isomorphism)

$$\alpha : \mathcal{O}^k_U \;\to\; \mathcal{F}|_U\,.$$

$\mathcal{F}$ is called of *finite presentation* when, for every $\xi \in X$, there is an open neighbourhood U and an exact sequence

$$\mathcal{O}^l_U \;\to\; \mathcal{O}^k_U \;\to\; \mathcal{F}|_U \;\to\; 0\,.$$

Remark 8.22. If $\alpha : \mathcal{O}^k_U \to \mathcal{F}|_U$ is an epimorphism, define $e_1 = (1, 0, \dots, 0), \dots, e_k = (0, \dots, 0, 1) \in \mathcal{O}^k_U$ and $f_1 = \alpha(U).e_1, \dots, f_k = \alpha(U).e_k \in \mathcal{F}(U)$; then $f_{1,x}, \dots, f_{k,x} \in \mathcal{F}_x$ generate $\mathcal{F}_x$ over $\mathcal{O}_{X,x}$ for every $x \in U$. Hence, $\mathcal{F}$ is of finite type iff, for every $\xi \in X$, there exist finitely many sections $f_1, \dots, f_k$ of $\mathcal{F}$ on some open U around ξ such that $f_{1,x}, \dots, f_{k,x}$ generate $\mathcal{F}_x$ for all $x \in U$.

Example 8.23. If $X = (\mathbb{C}, \mathcal{O})$ (where $\mathcal{O} = \mathcal{O}_\mathbb{C}$), $A = \{1/n : n \in \mathbb{N}^*\}$, and $\mathcal{F}$ is defined by

$$\mathcal{F}(U) = \{f \in \mathcal{O}(U) : f|_{U \cap A} = 0\}\,,$$

then $\mathcal{F}$ is not of finite type (**Exercise**).

Definition 8.24. A sheaf $\mathcal{F}$ is called *coherent* (or, more precisely, $\mathcal{O}_X$*-coherent*) when

(i) $\mathcal{F}$ is of finite type, and

(ii) for every open $U \subset X$ and every homomorphism $\mathcal{O}_U^k \to \mathcal{F}|_U$, the kernel is of finite type.

Remark 8.25. Condition (ii) above can be, equivalently, formulated as follows:
Given finitely many sections $f_1, \dots, f_k$ of $\mathcal{F}$ on some open $U \subset X$, the submodule

$$\big\{(r_1, \dots, r_k) \in \mathcal{O}_U^k(V) : r_1 f_1|_V + \dots + r_k f_k|_V = 0\big\}_{V \in \mathfrak{Top} U}$$

of $\mathcal{O}_U^k$ is of finite type.

Remark 8.26. If X is a ringed space and $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, then the *support* of $\mathcal{F}$

$$\mathrm{supp}\mathcal{F} = \{x \in X : \mathcal{F}_x \neq 0\} \subset X$$

is a closed set. Indeed, if $U \subset X$ is open and $f \in \mathcal{F}(U)$, then $\{x \in X : f_x = 0\} \subset U$ is clearly open. For $\xi \in X$, take an open neighbourhood U and sections $f_1, \dots, f_s \in \mathcal{F}(U)$ whose germs at x generate $\mathcal{F}_x$ for all $x \in U$. Then

$$\mathrm{supp}\mathcal{F} \cap U = \bigcup_{i=1}^{s} \{x \in U : (f_i)_x \neq 0\}$$

and the assertion follows.

Lemma 8.27. *Let X be a ringed space.*

1. *Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and $\mathcal{G} \subset \mathcal{F}$ a submodule. Then $\mathcal{G}$ is coherent if and only if it is of finite type.*
2. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of $\mathcal{O}_X$-modules. If two of the modules are coherent, then so is the third one.*
3. *If $\alpha : \mathcal{F} \to \mathcal{G}$ is a homomorphism of coherent $\mathcal{O}_X$-modules, then* $\ker \alpha$ *and* $\mathrm{coker}\, \alpha$ *are coherent $\mathcal{O}_X$-modules.*
4. *If $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules, then $\mathcal{F} \times \mathcal{G}$ and* $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ *are coherent $\mathcal{O}_X$-modules. In particular, $\mathcal{O}_X^n$ is $\mathcal{O}_X$-coherent if $\mathcal{O}_X$ is so.*

Proof. We will prove properties *1* and (most of) *2* here; the remainder of *2* (that will never be used in this course) can be found in [Se], and the rest is an easy **Exercise**.
For the proof of *1*, it suffices to show that, for every $U \in \mathfrak{Top} X$, $k \in \mathbb{N}$, and a homomorphism $\gamma : \mathcal{O}_U^k \to \mathcal{G}|_U$, the kernel of γ is of finite type. But $\ker \gamma = \ker(\iota \circ \gamma)$ (where $\iota : \mathcal{G} \hookrightarrow \mathcal{F}$) which is of finite type, by coherence of $\mathcal{F}$.

Now for *2*, suppose first that modules $\mathcal{F}$ and $\mathcal{F}''$ are coherent, ad an exact sequence $0 \to \mathcal{F}' \overset{\alpha}{\to} \mathcal{F} \overset{\beta}{\to} \mathcal{F}'' \to 0$ is given. By part *1*, it suffices to show that $\mathcal{F}'$ is of finite type. Let $\xi \in X$. By coherence of $\mathcal{F}$, there exists an open U around ξ and an epimorphism $\gamma : \mathcal{O}_U^k \to \mathcal{F}|_U$. By coherence of $\mathcal{F}''$, the kernel $\ker(\beta|_U \circ \gamma)$ is of finite type, and hence its surjective image $\ker \beta|_U = \gamma(\ker(\beta|_U \circ \gamma))$ is also of finite type. This shows that $\mathcal{F}'_U \cong \ker \beta|_U$ is of finite type, and so there is an open $V \subset U$ around ξ, $l \in \mathbb{N}$, and an epimorphism $\mathcal{O}_V^l \to \mathcal{F}'|_V$, as required.

Finally suppose that $\mathcal{F}'$ and $\mathcal{F}$ are coherent, and an exact sequence $0 \to \mathcal{F}' \overset{\alpha}{\to} \mathcal{F} \overset{\beta}{\to} \mathcal{F}'' \to 0$ is given. Then $\mathcal{F}''$ is of finite type, as a surjective image of a coherent $\mathcal{F}$. Let then open $U \subset X$, $k \in \mathbb{N}$,

and a homomorphism $\gamma : \mathcal{O}_U^k \to \mathcal{F}''$ be given. Write $h_1 = \gamma(e_1), \dots, h_k = \gamma(e_k) \in \mathcal{F}''$. Let $\xi \in U$. By surjectivity of β, there is a neighbourhood $V \subset U$ of ξ, and sections $f_1, \dots, f_k \in \mathcal{F}(V)$ such that $h_i = \beta(f_i)$, $i = 1, \dots, k$. Also, $\mathcal{F}'$ being of finite type, there exist (after shrinking V, if necessary) sections $g_1, \dots, g_q \in \mathcal{F}'(V)$ such that the germs $g_{1,x}, \dots, g_{q,x}$ generate $\mathcal{F}'_x$ for every $x \in V$.
Now, for $(r_1, \dots, r_k) \in \mathcal{O}_U^k(V)$, we have

$$\sum_{i=1}^k r_i h_i = 0 \ \Leftrightarrow \ 0 = \sum_{i=1}^k r_i \beta(f_i) = \beta(\sum_{i=1}^k r_i f_i)\,,$$

since β is a homomorphism of $\mathcal{O}_V$-modules. In other words, $\sum_{i=1}^k r_i h_i = 0$ iff $\sum_{i=1}^k r_i f_i \in \ker \beta$. Hence

$$\sum_{i=1}^k r_i h_i = 0 \quad \Leftrightarrow \quad \exists s_1, \dots, s_q \in \mathcal{O}_V \ : \ \sum_{i=1}^k r_i f_i = \sum_{j=1}^q s_j \alpha(g_j)\,.$$

The $f_1, \dots, f_k, \alpha(g_1), \dots, \alpha(g_q)$ being sections of a coherent $\mathcal{F}|_V$, the module

$$M \ = \ \Big\{(a_1, \dots, a_{k+q}) \in \mathcal{O}_V^{k+q}(W) \ : \ a_1 f_1 + \dots + a_k f_k + a_{k+1}\alpha(g_1) + \dots + a_{k+q}\alpha(g_q) = 0\Big\}_{W \in \mathfrak{Top} U}$$

is of finite type. But the module we are looking for is just the image of M under the canonical projection $\pi : \mathcal{O}_V^{k+q} \to \mathcal{O}_V^k$, and hence of finite type as well. □

Corollary 8.28. *Let X be a ringed space.*

1. *If $\mathcal{O}_X$ is $\mathcal{O}_X$-coherent, then every $\mathcal{O}_X$-module of finite presentation is $\mathcal{O}_X$-coherent. (The converse always holds.)*
2. *Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of coherent $\mathcal{O}_X$-modules. If $\alpha_\xi : \mathcal{F}_\xi \to \mathcal{G}_\xi$ is a monomorphism [epimorphism, isomorphism], then $\alpha|_U$ is so on some open neighbourhood U of ξ.*
3. *Let $\mathcal{F}, \mathcal{F}' \subset \mathcal{G}$ be coherent submodules. If $\mathcal{F}'_\xi \subset \mathcal{F}_\xi$ for $\xi \in X$, then $\mathcal{F}'|_U \subset \mathcal{F}|_U$ on some open neighbourhood U of ξ.*

Proof. **Exercise**. □

Lemma 8.29. *Let X be a ringed space, and assume $\mathcal{O}_X$ is $\mathcal{O}_X$-coherent. If $\mathcal{I} \subset \mathcal{O}_X$ is a coherent ideal, and $\mathcal{F}$ is an $\mathcal{O}_X/\mathcal{I}$-module, then $\mathcal{F}$ is $\mathcal{O}_X/\mathcal{I}$-coherent if and only if $\mathcal{F}$ is $\mathcal{O}_X$-coherent. In particular, $\mathcal{O}_X/\mathcal{I}$ is $\mathcal{O}_X/\mathcal{I}$-coherent.*

Proof. From the canonical exact sequence $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0$ and Lemma 8.27.2, it follows that $\mathcal{O}_X/\mathcal{I}$ is $\mathcal{O}_X$-coherent. Clearly, every $\mathcal{O}_X/\mathcal{I}$-module is of finite type with respect to $\mathcal{O}_X$ if and only if it is of finite type with respect to $\mathcal{O}_X/\mathcal{I}$.

Suppose $\mathcal{F}$ is $\mathcal{O}_X$-coherent, and let an exact sequence

$$0 \to \mathcal{K} \to ((\mathcal{O}_X/\mathcal{I})|_U)^k \to \mathcal{F}|_U$$

over an open $U \subset X$ be given. Then $\mathcal{K}$ is $\mathcal{O}_X$-coherent, as the kernel of a morphism of $\mathcal{O}_X$-coherent sheaves. In particular, $\mathcal{K}$ is of finite type with respect to $\mathcal{O}_X$, hence of finite type with respect to $\mathcal{O}_X/\mathcal{I}$.

Suppose now that $\mathcal{F}$ is $\mathcal{O}_X/\mathcal{I}$-coherent and let an exact sequence

$$0 \to \mathcal{K} \to (\mathcal{O}_X|_U)^k \to \mathcal{F}|_U$$

over an open $U \subset X$ be given. Moding out $\mathcal{I}$, we get again an exact sequence $0 \to \mathcal{K}/\mathcal{I} \to ((\mathcal{O}_X/\mathcal{I})|_U)^k \to \mathcal{F}|_U$, hence $\mathcal{K}/\mathcal{I}$ is $\mathcal{O}_X/\mathcal{I}$-coherent, as the kernel of a morphism of $\mathcal{O}_X/\mathcal{I}$-coherent sheaves. In particular, $\mathcal{K}/\mathcal{I}$ is of finite type with respect to $\mathcal{O}_X/\mathcal{I}$, hence of finite type over $\mathcal{O}_X$. Now, $\mathcal{I}$ and $\mathcal{K}/\mathcal{I}$ being of finite type with respect to $\mathcal{O}_X$, so must be $\mathcal{K}$, which completes the proof. □

Finally, to conclude we list two simple yet very useful corollaries that allow the passage from local to semi-global in coherent sheaves.

Proposition 8.30. *Let $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, and $\xi \in X$ be given. Let $\rho : (\mathrm{Hom}(\mathcal{F},\mathcal{G}))_\xi \to \mathrm{Hom}(\mathcal{F}_\xi,\mathcal{G}_\xi)$ be the canonical homomorphism.*

(i) If $\mathcal{F}$ is of finite type, then ρ is injective.

(ii) If $\mathcal{F}$ is of finite presentation, then ρ is an isomorphism.

Proof. **Exercise** (see, e.g., [Se]). □

From the above proposition and Lema 8.27.3, we get the following:

Corollary 8.31. *Let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules, $\xi \in X$, and let $\beta : \mathcal{F}_\xi \to \mathcal{G}_\xi$ be a monomorphism (resp. epimorphism). Then there is an open neighbourhood U of ξ and a monomorphism (resp. epimorphism) $\alpha : \mathcal{F}|_U \to \mathcal{G}|_U$ such that $\alpha_\xi = \beta$.*

Example 8.32. The above does not hold in general. Consider, e.g., the sheaf $\mathcal{F}$ of Example 8.23. We have $\mathcal{F}_0 = \{0\} \overset{\cong}{\to} \underline{0}_0$, where $\underline{0}$ is the zero sheaf on $\mathbb{C}$, but $\mathcal{F}_x$ doesn't embed into $\underline{0}_x$ for any $x \neq 0$.

From Corolary 8.31 and Proposition 8.30 again:

Corollary 8.33. *Let X be a ringed space such that $\mathcal{O}_X$ is coherent, and let $\xi \in X$.*

(i) If M is an $\mathcal{O}_{X,\xi}$-module of finite presentation, then there are an open neighbourhood U of ξ, and a coherent $\mathcal{O}_U$-module $\mathcal{G}$, such that $\mathcal{G}_\xi = M$.

(ii) If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, and $M \subset \mathcal{F}_\xi$ is a finitely generated submodule, then there are an open neighbourhood U of ξ, and a coherent $\mathcal{O}_U$-submodule $\mathcal{G} \subset \mathcal{F}|_U$, such that $\mathcal{G}_\xi = M$.

9 Complex analytic spaces

9.1 Oka coherence theorem

We present here the fundamental theorem of Oka asserting that the sheaf $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n}$ of holomorphic functions in $\mathbb{C}^n$ is $\mathcal{O}$-coherent. In fact, for the purpose of having a stronger inductive hypothesis, we prove slightly more.

Definition 9.1. Let $\Omega \subset \mathbb{C}^n$ be open, $q \in \mathbb{N}_+$, and let $f_i : \Omega \to \mathbb{C}^q$ be holomorphic mappings, $i = 1\dots,p$. For every open $U \subset \Omega$, consider the submodule $\mathcal{R}^{f_1\cdots f_p}(U) \subset (\mathcal{O}(U))^p$ defined as

$$\mathcal{R}^{f_1\cdots f_p}(U) = \{(c_1,\dots,c_p) \in (\mathcal{O}(U))^p : \sum_{i=1}^{p} c_i f_i \equiv 0\}\,.$$

The family $\{\mathcal{R}^{f_1\cdots f_p}(U)\}_{U\in\mathfrak{Top}\Omega}$ forms a submodule of $\mathcal{O}^p_\Omega$, called the *sheaf of relations among* f_1, $\dots, f_p$.

Theorem 9.2 (Oka Coherence)**.** *Given an open $\Omega \subset \mathbb{C}^n$ and holomorphic mappings $f_i : \Omega \to \mathbb{C}^q$ ($i = 1,\dots,p$), the sheaf of relations among $f_1,\dots,f_p$ is of finite type.*

Proof. We will proceed by induction on n. If $n = 0$, then $\mathbb{C}^n$ (as well as all its nonempty open subsets) is the singleton $\{0\}$. Consequently, for any $q \in \mathbb{N}$, the f_i are constant functions, and $\mathcal{R}^{f_1\cdots f_p}$ is a finite-dimensional vector subspace of $\mathbb{C}^p$, hence trivially of finite type. The rest of the proof is divided into two steps: First, we shall show that, for given $n \geq 1$, if the theorem holds for $q = 1$, then it holds for all values of q. Secondly, if the theorem holds for $n-1$ and all q, then it holds for n and $q = 1$.

Step 1. Induction on q. The case $q = 1$ being assumed true, suppose that $q > 1$ and $f_1,\dots,f_p$ are holomorphic in some open Ω around $\xi = 0$ in $\mathbb{C}^n$, with values in $\mathbb{C}^q$. We want to show that there is an open $V \subset \Omega$ around 0 and a finite number of sections of $\mathcal{R}^{f_1\cdots f_p}(V)$ whose germs at x generate $\mathcal{R}^{f_1\cdots f_p}_x$ for every $x \in V$ (the same argument will, of course, apply to any other point of Ω).
Notice that every relation $\sum_i c_i f_i \equiv 0$ is equivalent to the conjunction of relations

$$\sum_i c_i g_i \equiv 0 \quad \text{and} \quad \sum_i c_i h_i \equiv 0\,,$$

where $f_i = (g_i, h_i)$, $g_i : \Omega \to \mathbb{C}^{q-1}$, $h_i : \Omega \to \mathbb{C}$, $i = 1,\dots,p$. By the inductive hypothesis, the module $\mathcal{R}^{g_1\cdots g_p}$ is of finite type, hence there are an open neighbourhood U of the origin, and a finite number of p-tuples $\{(d^j_1,\dots,d^j_p)\}_{j=1}^s$ from $\mathcal{R}^{g_1\cdots g_p}(U)$, such that the germs $(d^j_1,\dots,d^j_p)_x$ generate $\mathcal{R}^{g_1\cdots g_p}_x$ at every $x \in U$. Consider now

$$k_j = \sum_{i=1}^{p} d^j_i h_i\,, \quad j = 1,\dots,s\,,$$

and the submodule $\mathcal{R}^{k_1\cdots k_s}$ of $(\mathcal{O}_U)^s$. As the k_j are $\mathbb{C}$-valued functions, we have by assumption ($q = 1$) that $\mathcal{R}^{k_1\cdots k_s}$ is of finite type. Therefore there are an open $V \subset U$ around 0 and sections $\{(\lambda^l_1,\dots,\lambda^l_s)\}_{l=1}^t$ of $\mathcal{R}^{k_1\cdots k_s}(V)$, such that the germs $(\lambda^l_1,\dots,\lambda^l_s)_x$ generate $\mathcal{R}^{k_1\cdots k_s}_x$ for every $x \in V$.

Let $(c_1,\dots,c_p)$ be an arbitrary section of $\mathcal{R}^{f_1\cdots f_p}(\Omega)$. Since $\mathcal{R}^{f_1\cdots f_p}$ is a submodule of $\mathcal{R}^{g_1\cdots g_p}$, then, for every $x \in U$,

$$(c_1,\dots,c_p)_x = \sum_{j=1}^{s} \kappa^x_j (d^j_1,\dots,d^j_p)_x = (\sum_j \kappa^x_j d^j_{1,x},\dots,\sum_j \kappa^x_j d^j_{p,x})\,,$$

for some $\kappa_j^x \in \mathcal{O}_x$. Since $\mathcal{R}^{f_1 \dots f_p}$ is a submodule of $\mathcal{R}^{h_1 \dots h_p}$, then

$$\sum_{i=1}^{p} \sum_{j=1}^{s} \kappa_j^x d_{i,x}^j h_{i,x} = 0 \quad \text{in} \quad \mathcal{O}_x\,, \qquad \text{for all } x \in U\,.$$

Hence, for all $x \in U$, $(\kappa_1^x, \dots, \kappa_s^x) \in \mathcal{R}_x^{k_1 \dots k_s}$, and thus, for all $x \in V$,

$$(\kappa_1^x, \dots, \kappa_s^x) = \sum_{l=1}^{t} \eta_l^x (\lambda_1^l, \dots, \lambda_s^l)_x = (\sum_l \eta_l^x \lambda_{1,x}^l, \dots, \sum_l \eta_l^x \lambda_{s,x}^l)\,.$$

Therefore, for all $x \in V$,

$$(c_1, \dots, c_p)_x = (\sum_{j,l} \eta_l^x \lambda_{j,x}^l d_{1,x}^j, \dots, \sum_{j,l} \eta_l^x \lambda_{j,x}^l d_{p,x}^j) = \sum_l \eta_l^x (\sum_j \lambda_j^l d_1^j, \dots, \sum_j \lambda_j^l d_p^j)_x\,,$$

which is to say that the finitely many p-tuples $\{(\sum_j \lambda_j^l d_1^j, \dots, \sum_j \lambda_j^l d_p^j)\}_{l=1}^t$ from $\mathcal{R}^{f_1 \dots f_p}(V)$ generate $\mathcal{R}_x^{f_1 \dots f_p}$ at every $x \in V$, as required.

Step 2. Suppose the theorem holds for $n-1 \geq 0$ and all q, and let $f_1, \dots, f_p \in \mathcal{O}(\Omega)$ for some open $\Omega \subset \mathbb{C}^n$ around the origin. (As above, we will only prove the assertion at the origin, the proof being exactly the same at any other point.) Without loss of generality, we may assume that none of the f_i is constant in Ω, and hence, by Remark 4.8, that all the germs $(f_1)_0, \dots, (f_p)_0$ are regular in x_n. Put $z = (x_1, \dots, x_{n-1})$ and $w = x_n$, for simplicity of notation. By Weierstrass Preparation, after shrinking $\Omega = U \times \Delta$ if necessary, we have, for each $i = 1, \dots, p$, a unique pair (P_i, q_i) such that $f_i = q_i P_i$, where $q_i \in \mathcal{O}(\Omega)$, $q_i(z,w) \neq 0$ for $(z,w) \in \Omega$, and $P_i \in \mathcal{O}(U)[w]$ is monic. (If $f_i(0) \neq 0$, we put $P_i \equiv 1$.) Now, clearly, $\mathcal{R}^{f_1 \dots f_p}$ is of finite type iff $\mathcal{R}^{P_1 \dots P_p}$ is of finite type, so it suffices to prove the latter claim. Let $\alpha = \deg P_p$; we may assume that $\alpha \geq \deg P_i$ for all i. Let T denote the set of $p-1$ p-tuples $(c_1, \dots, c_p)$ of the form:

$$\begin{cases} c_1 = -P_p\,, \quad c_p = P_1, \quad c_i = 0 \ \text{ for } \ i \notin \{1,p\} \\ c_2 = -P_p\,, \quad c_p = P_2, \quad c_i = 0 \ \text{ for } \ i \notin \{2,p\} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ c_{p-1} = -P_p\,,\ c_p = P_{p-1},\ c_i = 0 \ \text{ for } \ i \notin \{p-1,p\}\,. \end{cases}$$

Then $(c_1, \dots, c_p) \in \mathcal{R}^{P_1 \dots P_p}(\Omega)$ for every $(c_1, \dots, c_p) \in T$. For the proof of Theorem 9.2, we will need the following auxillary proposition.

Proposition 9.3. *Let $P_1, \dots, P_p \in \mathcal{O}(U)[w]$ be monic, of degrees at most $\alpha = \deg P_p$, as above. Then, at every point $(a,b) \in U \times \Delta$, every germ $(c_1, \dots, c_p)_{(a,b)} \in \mathcal{R}_{(a,b)}^{P_1 \dots P_p}$ is a linear combination (over $\mathcal{O}_{(a,b)}$) of:*
(1) germs of p-tuples of the set T, and
(2) elements of $\mathcal{R}_{(a,b)}^{P_1 \dots P_p} \cap (\mathcal{O}_a[w])^p$, all of whose components are of degree at most $\alpha - 1$.

Assume for the moment the above proposition. By finiteness of T, it now suffices to show that the sheaf

$$\{(c_1, \dots, c_p) \in \mathcal{R}^{P_1 \dots P_p}(U) \cap (\mathcal{O}(U)[w])^p \ : \ \deg c_i \leq \alpha - 1\}_{U \in \mathfrak{Top}\Omega}$$

is of finite type. This however follows from the assumptions for $n-1$, since every relation $\sum_i c_i P_i \equiv 0$ is equivalent to

$$\sum_{j=1}^{p\alpha} d_j Q_j \equiv 0\,,$$

where $c_i(z,w) = c_i^1(z)w^{\alpha-1} + \cdots + c_i^{\alpha-1}(z)w + c_i^{\alpha}(z)$, $P_i(z,w) = P_i^0(z)w^{\alpha} + \cdots + P_i^{\alpha-1}(z)w + P_i^{\alpha}(z)$, $d_{\alpha(k-1)+l} = c_k^l \in \mathcal{O}(U)$, and the components of the $Q_j \in (\mathcal{O}(U))^q$ are combinations of the P_i^l.

It thus remains to prove Proposition 9.3. Let U and $P_1, \ldots, P_p$ be as above, and let $(a,b) \in U \times \Delta$. Since the germ $(P_p)_{(a,b)}$ is regular in w, from Weierstrass Preparation again, we get that

$$P_p = P' \cdot P''$$

in an open $V \times B_r(b)$ around (a,b), where both $P', P'' \in \mathcal{O}(V)[w]$, $P''(a,b) \neq 0$, and $P'_{(a,b)}$ is distinguished in $w - b$.

Consider $(c_1, \ldots, c_p) \in \mathcal{R}^{P_1 \ldots P_p}(V \times B_r(b))$. After shrinking V and r if necessary, we get by Weierstrass Division,

$$c_i = \mu_i P' + c_i'\,, \qquad i = 1, \ldots, p-1\,,$$

where $\mu_i \in \mathcal{O}(V \times B_r(b))$, and the $c_i' \in \mathcal{O}(V)[w]$ are of degrees strictly less than $\deg P'$. Put

$$c_p' = c_p + \sum_{i \leq p-1} \frac{\mu_i}{P''} P_i\,.$$

Then, modulo T, $\sum_i c_i P_i \equiv 0$ if and only if $\sum_i c_i' P_i \equiv 0$, and further, if and only if $\sum_i (P'' c_i') P_i \equiv 0$, as $P''(a,b) \neq 0$. We claim that all the $P'' c_i'$ are polynomials in w. This is clearly true for all $i \leq p-1$, and for $i = p$ we have

$$P'' c_p' \;=\; P'' c_p + P'' \sum_{i \leq p-1} \frac{\mu_i}{P''} P_i \;=\; \frac{-1}{P'} \sum_{i \leq p-1} c_i P_i + \sum_{i \leq p-1} \mu_i P_i \;=\; \frac{-1}{P'} \cdot \sum_{i \leq p-1} c_i' P_i\,.$$

By Weierstrass Division by P' applied to the polynomial $-(\sum_{i \leq p-1} c_i' P_i)$, we get

$$-\sum_{i \leq p-1} c_i' P_i = P'Q + R, \quad \text{where } \deg R < \deg P'\,,$$

hence $P'' c_p' = Q + \dfrac{R}{P'}$. This shows that the quotient R/P' is holomorphic in a neighbourhood of (a,b). But $w = b$ is the only root of $P'(a,\cdot)$ and $\deg P' > \deg R$, which is only possible when $R \equiv 0$; i.e., $P'' c_p' = Q$ is a polynomial as well.

We have thus shown that every relation $(c_1, \ldots, c_p)$ among $P_1, \ldots, P_p$ is, in a neighbourhood of (a,b), congruent modulo T to a relation $(P'' c_1', \ldots, P'' c_p')$ whose all components are polynomials with respect to w. To complete the proof of the proposition, it remains to show that every p-tuple $(c_1, \ldots, c_p) \in \mathcal{R}^{P_1 \ldots P_p}(V \times B_r(b)) \cap \mathcal{O}(V)[w]$ is (in a neighbourhood of (a,b)) congruent modulo T to a relation $(c_1'', \ldots, c_p'')$ among $P_1, \ldots, P_p$, whose all components are polynomials in w of degrees strictly less than $\alpha = \deg P_p$. By Weierstrass Division again, we have

$$c_i = \nu_i P_p + c_i'', \quad 1 \leq i \leq p-1\,,$$

where $\nu_i \in \mathcal{O}(V)[w]$, and $c_i'' \in \mathcal{O}(V)[w]$ are of degrees at most $\alpha - 1$. Put

$$c_p'' = c_p + \sum_{i \le p-1} \nu_i P_i \,.$$

Then $(c_1, \ldots, c_p)$ is congruent modulo T to $(c_1'', \ldots, c_p'')$. All the $c_1'', \ldots, c_{p-1}''$ are polynomials in w of degrees at most $\alpha - 1$, and as $\nu_1, \ldots, \nu_{p-1}$ are also polynomials in w, then so is c_p''. Finally, it follows from

$$c_p'' P_p = -(\sum_{i \le p-1} c_i'' P_i)$$

that $\deg(c_p'' P_p) \le 2\alpha - 1$, and hence $\deg c_p'' \le \alpha - 1$ ($\deg(c_p'' P_p) = \deg c_p'' + \deg P_p$, as P_p is of degree α in $w - b$). This completes the proof of Proposition 9.3, and hence also of Theorem 9.2. □

Corollary 9.4. *The sheaf $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n}$ of holomorphic functions is $\mathcal{O}$-coherent.*

By Lemma 8.27.1 and Corollary 8.28.1, we now obtain immediately the

Corollary 9.5. *(a) Every sheaf of ideals $\mathcal{I} \subset \mathcal{O}$ of finite type is coherent.*

(b) Every $\mathcal{O}$-module $\mathcal{F}$ of finite presentation is coherent.

Corollary 9.6. *The intersection $\mathcal{I}_1 \cap \cdots \cap \mathcal{I}_s$ of coherent $\mathcal{O}$-ideals is coherent.*

Proof. It suffices to prove the result for $s = 2$. Since $\mathcal{I}_1 + \mathcal{I}_2$ is of finite type (as $\mathcal{I}_1$ and $\mathcal{I}_2$ are so) and hence, by Corollary 9.5 above, $\mathcal{I}_1 + \mathcal{I}_2$ is coherent, so is the quotient $(\mathcal{I}_1 + \mathcal{I}_2)/\mathcal{I}_2$. Therefore also $\mathcal{I}_1 \cap \mathcal{I}_2$ is coherent, by exactness of the canonical sequence

$$0 \to \mathcal{I}_1 \cap \mathcal{I}_2 \to \mathcal{I}_1 \to \frac{\mathcal{I}_1 + \mathcal{I}_2}{\mathcal{I}_2} \to 0\,.$$

□

Corollary 9.7. *Suppose $\Omega \Subset \mathbb{C}^n$ is open, $a \in \Omega$, and functions $g_1, \ldots, g_p, g \in \mathcal{O}(\Omega)$ have the following property:*

$$f \in \mathcal{O}(V),\ a \in V \subset \Omega,\ (gf)_a \in \sum_{i=1}^{p} \mathcal{O}_{\Omega,a} \cdot g_{i,a} \quad \Longrightarrow \quad f_a \in \sum_{i=1}^{p} \mathcal{O}_{\Omega,a} \cdot g_{i,a}\,. \tag{9.1}$$

Then there is an open neighbourhood $U \subset \Omega$ of a such that, for every $x \in U$ and every function h holomorphic in a neighbourhood of x, $(gh)_x \in \sum_1^p \mathcal{O}_{\Omega,x} \cdot g_{i,x}$ implies $h_x \in \sum_1^p \mathcal{O}_{\Omega,x} \cdot g_{i,x}$.

Proof. By Oka's Theorem 9.2, the sheaf $\mathcal{R}^{g_1 \cdots g_p g}$ of relations among $g_1, \ldots, g_p, g$ is of finite type, hence there is a neighbourhood $V \subset \Omega$ of a and finitely many

$$(s_1^1, \ldots, s_p^1, f^1), \ldots, (s_1^q, \ldots, s_p^q, f^q) \in \mathcal{R}^{g_1 \cdots g_p g}(V)$$

whose germs at x generate $\mathcal{R}_x^{g_1 \cdots g_p g}$ for all $x \in V$. The property (9.1) now implies that $f_a^1, \ldots, f_a^q \in \sum_1^p \mathcal{O}_{\Omega,a} \cdot g_{i,a}$, and hence there is an open $U \subset V$ around a and functions $f_i^j \in \mathcal{O}(U)$ such that $f^j = \sum_1^p f_i^j g_i$ on U, $j = 1, \ldots, q$.

Therefore, if h is a function holomorphic in a neighbourhood of a point $x \in U$ satisfying $(gh)_x \in \sum_1^p \mathcal{O}_{\Omega,x} \cdot g_{i,x}$, then $(s_1', \ldots, s_p', h)_x \in \mathcal{R}_x^{g_1 \cdots g_p g}$ for some $s_1', \ldots, s_p'$ holomorphic in a neighbourhood of x, hence h_x is a combination of $f_x^1, \ldots, f_x^q$, and hence a combination of the $g_{i,x}$, as required. □

9.2 Cartan coherence theorem

Definition 9.8. Let $\Omega \subset \mathbb{C}^n$ be open. For an analytic set $A \subset \Omega$, we define the *(full) sheaf of ideals of* A, denoted $\mathfrak{J}_A$, as

$$\mathfrak{J}_A(U) = \{f \in \mathcal{O}_\Omega(U) : f|_{A\cap U} = 0\} \qquad \text{for } \; U \in \mathfrak{Top}\,\Omega\,.$$

Theorem 9.9 (Cartan Coherence)**.** *If $\Omega \subset \mathbb{C}^n$ is open and $A \subset \Omega$ analytic, then the sheaf $\mathfrak{J}_A$ is $\mathcal{O}_\Omega$-coherent.*

Proof. Let $\Omega \subset \mathbb{C}^n$ be open and nonempty, and let $A \subset \Omega$ be a proper analytic subset. By Corollary 9.5, it suffices to show that $\mathfrak{J}_A$ is of finite type with respect to $\mathcal{O}_\Omega$. Let a be a point in Ω. Without loss of generality, we may assume that $a = 0$ and $a \in A$, for otherwise, by openness of $\Omega \setminus A$, there is an open neighbourhood U of a in Ω for which $U \cap A = \varnothing$, and hence $\mathfrak{J}_A(U) = \mathcal{O}_\Omega(U)$, as $1 \in \mathfrak{J}_A(U)$. Next observe that we may assume A to be irreducible at a. Indeed, by Proposition 4.18, $A_a = A^1_a \cup \dots \cup A^s_a$ is the union of finitely many irreducible analytic germs, hence $A = A^1 \cup \dots \cup A^s$ in some neighbourhood U of a, where A^j are representatives of the respective A^j_a, irreducible at a. Since $(\mathfrak{J}_A)_x = (\mathfrak{J}_{A^1})_x \cap \dots \cap (\mathfrak{J}_{A^s})_x$ for all $x \in U$, it suffices to know that the intersection of coherent $\mathcal{O}_\Omega$-ideals is itself coherent, which is Corollary 9.6.

Let then A be irreducible at a, of dimension k, say. By Ideal Normalization (Corollary 6.5), $(\mathfrak{J}_A)_a = \{f_a \in \mathcal{O}_{\Omega,a} : f_a|_{A_a} = 0\}$ is k-regular (after an analytic change of coordinates at a, if necessary). Hence by the proof of Rückert Lemma 6.8, there are a system of coordinates $z = (x_1, \dots, x_k)$ at 0 in $\mathbb{C}^k$, an open connected neighbourhood V of $0 \in \mathbb{C}^k$, and $r > 0$, defining a neighbourhood $U = V \times r\Delta^{n-k}$ of the origin $a \in \mathbb{C}^n$, with the following properties:

1° if $W \subset U$ and a function $f \in \mathcal{O}(W) \cap \mathfrak{J}_A(W)$ depends only on variables z, then $f \equiv 0$;

2° there exists a monic irreducible polynomial $F \in \mathcal{O}(V)[w]$ such that

$$F(z, x_{k+1}) \in \mathfrak{J}_A(U)\,,$$

and the discriminant $\delta = D(F) \in \mathcal{O}(V)$ satisfies

$$\delta_x \notin (\mathfrak{J}_A)_x \quad \text{for all } x \in U \qquad (\text{by } 1°)\,;$$

3° there are monic irreducible $Q_{k+2}, \dots, Q_n \in \mathcal{O}(V)[w]$ such that

$$Q_{k+j}(z, x_{k+j}) \in \mathfrak{J}_A(U) \quad \text{for } j = 2, \dots, n-k\,;$$

4° there are polynomials $P_{k+2}, \dots, P_n \in \mathcal{O}(V)[w]$ such that

$$\delta(z)x_{k+j} - P_{k+j}(z, x_{k+1}) \in \mathfrak{J}_A(U) \quad \text{for } j = 2, \dots, n-k\,;$$

5° every point $b \in A \cap U$ lies in the closure of a k-dimensional subvariety of U defined by equations

$$F(z, x_{k+1}) \;=\; \delta(z)x_{k+2} - P_{k+2}(z, x_{k+1}) \;=\; \dots \;=\; \delta(z)x_n - P_n(z, x_{k+1}) \;=\; 0\,, \qquad (9.2)$$

for $z \in V \setminus \delta^{-1}(0)$. Moreover, for every $z \in V \setminus \delta^{-1}(0)$ and every root w of $F(z, \cdot)$, there is at least one point in $A \cap U$ whose first $k+1$ coordinates are (z, w) (by (9.2)).

Let $\mathcal{S}$ denote a finite set of functions from $\mathcal{O}(U)$, including those of 2°, 3° and 4°, whose germs at a generate the prime ideal $(\mathfrak{J}_A)_a$ (in fact, it suffices to take *only* those of 2° and 4°, cf. Proposition **??**). We will denote by $\mathcal{S}_x$ the collection of germs at $x \in U$ of the elements of $\mathcal{S}$. We shall first prove the following

Lemma 9.10. *There is an open neighbourhood U of the origin $a \in \mathbb{C}^n$ with the following property: if $b \in U$, and h is a function holomorphic in a neighbourhood of b such that $(\delta^k h)_b \in \mathcal{S}_b$ for some $k \in \mathbb{N}$, then $h_b \in \mathcal{S}_b$.*

Proof. Indeed, apply Corollary 9.7 with $g_1, \ldots, g_p$ the elements of $\mathcal{S}$ and $g = \delta$. Then, the property (9.1) of the Corollary is satisfied, since $\mathcal{S}_a = (\mathfrak{J}_A)_a$ is prime, and $\delta_a \notin (\mathfrak{J}_A)_a$. Hence, after shrinking U if necessary, we can conclude that, for every $b \in U$ and h holomorphic in a neighbourhood of b, $(\delta^k h)_b \in \mathcal{S}_b$ implies $(\delta^{k-1} h)_b \in \mathcal{S}_b$. The result now follows by induction on k. □

To complete the proof of Theorem 9.9, it suffices to show that $(\mathfrak{J}_A)_b$ is generated by the elements of $\mathcal{S}_b$ for all $b \in U$. Let then $b \in A \cap U$, and let f be a function holomorphic in a neighbourhood of b, that vanishes on A. Consider the germs $F_b, (Q_{k+2})_b, \ldots, (Q_n)_b \in \mathcal{S}_b$, and let $z_b = (b_1, \ldots, b_k)$, where $b = (b_1, \ldots, b_k, \ldots, b_n)$. As in the proof of Proposition 9.3, since F is monic in x_{k+1}, the germ F_b is regular in $x_{k+1} - b_{k+1}$, and similarly the $(Q_{k+j})_b$ are regular in $x_{k+j} - b_{k+j}$, $j = 2, \ldots, n-k$. Therefore, after evaluation at $z = z_b$, the series $F_b(z_b) \in \mathbb{C}\{x_{k+1}\}$, $(Q_{k+2})_b(z_b) \in \mathbb{C}\{x_{k+2}\}, \ldots, (Q_n)_b(z_b) \in \mathbb{C}\{x_n\}$ are nonzero, and consequently, the remainder of the Hironaka division

$$f_b = q_1 F_b + \sum_{i=2}^{n-k} q_i (Q_{k+i})_b + R_b$$

is finitely supported; i.e., $R_b \in \mathbb{C}[x_{k+1}, \ldots, x_n]$.

Let $d = \deg R_b$. Then, as in the proof of Rückert Lemma 6.8, $\delta_b^d R_b$ (hence also $(\delta^d f)_b$) is congruent modulo $(\delta(z)x_{k+2} - P_{k+2}(z, x_{k+1}))_b, \ldots, (\delta(z)x_n - P_n(z, x_{k+1}))_b \in \mathcal{S}_b$ to a polynomial G in x_{k+1} with coefficients holomorphic in a neighbourhood of z_b. Now, F_b being regular in $x_{k+1} - b_{k+1}$, it is an associate of a polynomial F'_b whose germ at b is distinguished in $x_{k+1} - b_{k+1}$. Hence $F'_b \in \mathcal{S}_b$, and we can divide (in $\mathcal{O}_k[x_{k+1} - b_{k+1}]$):

$$(\delta^k f)_b = q F'_b + H_b, \quad \text{where } H \in \mathcal{O}(W)[x_{k+1}] \text{ and } \deg H < \deg F'$$

for some open W around z_b.
Observe that, since f vanishes on $A \cap U'$ for some open $U' = b + r\Delta^n$ around b, and $(\delta^{-1}(0) \times \mathbb{C}^{n-k}) \cap A \cap U'$ is nowhere dense in $A \cap U'$ (Proposition 7.6), then, for z in an open dense set $(z_b + r\Delta^k) \setminus \delta^{-1}(0)$, $H(z, \cdot)$ vanishes whenever $F'(z, \cdot) = 0$, that is, at every root of $F'(z, \cdot)$, by 5°. As $\deg H < \deg F'$, it follows that $H \equiv 0$; i.e., $(\delta^d f)_b \in \mathcal{S}_b$. Therefore, by Lemma 9.10, $f_b \in \mathcal{S}_b$, which completes the proof. □

9.3 Complex analytic spaces - first properties

Lemma 9.11. *Let $\Omega \subset \mathbb{C}^n$ be an open subset, and assume that $\mathcal{I}$ is a coherent sheaf of $\mathcal{O}_\Omega$-ideals. Then*

$$\operatorname{supp}(\mathcal{O}_\Omega/\mathcal{I}) = \{x \in \Omega : (\mathcal{O}_\Omega/\mathcal{I})_x \neq 0\}$$

is an analytic subset of Ω.

Proof. For $\xi \in \Omega$, take an open neighbourhood $U \subset \Omega$ such that there exist $f_1, \ldots, f_s \in \mathcal{I}(U)$, whose germs at x generate $\mathcal{I}_x$ for every $x \in U$ (by coherence). Then, for $x \in U$, we have

$$\begin{aligned} x \in \operatorname{supp}(\mathcal{O}_\Omega/\mathcal{I}) \iff (\mathcal{O}_\Omega/\mathcal{I})_x \neq 0 \iff \mathcal{I}_x \neq \mathcal{O}_{\Omega,x} \iff \mathcal{I}_x \subset \mathfrak{m}_{\Omega,x} \\ \iff f_{1,x}, \ldots, f_{s,x} \in \mathfrak{m}_{\Omega,x} \iff f_1(x) = \cdots = f_s(x) = 0\,. \end{aligned}$$

Thus every $\xi \in \Omega$ admits an open neighbourhood U such that $\operatorname{supp}(\mathcal{O}_\Omega/\mathcal{I}) \cap U$ is defined by finitely many functions analytic on U, which proves the assertion. □

For a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, by the *radical* of $\mathcal{I}$, denoted rad$\mathcal{I}$, we understand the sheafification of the presheaf $\{\mathrm{rad}(\mathcal{I}(U))\}_{U \in \mathfrak{Top}X}$.

Corollary 9.12. *If $\Omega \subset \mathbb{C}^n$ is open and $\mathcal{I} \subset \mathcal{O}_\Omega$ is a coherent sheaf of ideals, then the sheaf* rad$\mathcal{I}$ *is coherent.*

Proof. By the above lemma, $A = \mathrm{supp}(\mathcal{O}_\Omega/\mathcal{I})$ is an analytic subset of Ω. Hence, by Nullstellensatz (Theorem 6.6), $(\mathrm{rad}\mathcal{I})_x = (\mathfrak{J}_A)_x$ for all $x \in \Omega$, so rad$\mathcal{I} = \mathfrak{J}_A$ is coherent, by Cartan's Coherence Theorem. □

Definition 9.13. Let $\Omega \subset \mathbb{C}^n$ be open, and assume a coherent ideal $\mathcal{I} \subset \mathcal{O}_\Omega$ is given. Then $A = \mathrm{supp}(\mathcal{O}_\Omega/\mathcal{I})$ is an analytic subset of Ω, and $(A, (\mathcal{O}_\Omega/\mathcal{I})|_A)$ is a $\mathbb{C}$-ringed space, which we call a *local model*.
A *complex analytic space* is a $\mathbb{C}$-ringed space $X = (|X|, \mathcal{O}_X)$ satisfying the following conditions:

(a) $|X|$ is Hausdorff

(b) For every $\xi \in X$, there is an open neighbourhood U of ξ such that $(U, \mathcal{O}_X|_U)$ is isomorphic (as $\mathbb{C}$-ringed space) to some local model.

If $X = (|X|, \mathcal{O}_X)$ and $Y = (|Y|, \mathcal{O}_Y)$ are complex analytic spaces, then any morphism

$$\varphi = (|\varphi|, \varphi^*) : (|X|, \mathcal{O}_X) \to (|Y|, \mathcal{O}_Y)$$

of $\mathbb{C}$-ringed spaces is called an *analytic map* (or *holomorphic map*).

A complex analytic space Y is called an *open complex analytic subspace* of X, if $|Y|$ is an open subset of $|X|$, and $\mathcal{O}_Y = \mathcal{O}_X|_Y$.
Y is called a *closed complex analytic subspace* of X, if there is a coherent ideal $\mathcal{I} \subset \mathcal{O}_X$ such that

$$|Y| = \mathrm{supp}(\mathcal{O}_X/\mathcal{I}) \quad \text{and} \quad \mathcal{O}_Y = (\mathcal{O}_X/\mathcal{I})|_Y .$$

In this case, there is a canonical analytic map determined by the injection, which we denote $Y \hookrightarrow X$.
A subset A of a complex analytic space X is called *analytic* when there is a coherent ideal $\mathcal{I} \subset \mathcal{O}_X$ such that

$$A = \mathrm{supp}(\mathcal{O}_X/\mathcal{I}) .$$

(Note that, if X is a manifold, then the above agrees with a standard definition of analytic set - **Exercise**!)
Finally, a space X is called *non-singular* (or *smooth*) at $\xi \in X$, if there is an open neighbourhood U of ξ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to some local model of the form $(\Omega, \mathcal{O}_\Omega)$, where $\Omega \subset \mathbb{C}^n$ is an open subset. Otherwise X is *singular* at ξ, and ξ is its *singular point.*

Example 9.14. Denote by z the coordinate function in $\mathbb{C}$, and let $\mathcal{I}_n \subset \mathcal{O}_\mathbb{C}$ be the sheaf of ideals generated by z^n ($n \in \mathbb{N}_+$). Then $\mathrm{supp}(\mathcal{O}_\mathbb{C}/\mathcal{I}_n) = \{0\}$, and $(\mathcal{O}_\mathbb{C}/\mathcal{I}_n)|_{\{0\}} \cong \mathbb{C}^n$ is an n-dimensional $\mathbb{C}$-vector space. The space $(\{0\}, \mathbb{C}^n)$ is called an *n-fold point*; for $n > 1$ it is singular.

As a consequence of Oka Coherence we obtain (**Exercise**) the fundamental:

Theorem 9.15. *The structure sheaf $\mathcal{O}_X$ of every complex analytic space X is coherent.*

Theorem 9.16. *If A is an analytic subset of an analytic space X, then the singular locus* $\mathrm{sng}A$ *forms an analytic subset of A. In particular,* $\mathrm{sng}X$ *is analytic.*

Proof. The problem being local, we may assume that X is a local model, and furthermore, that X is smooth (as analytic subsets of analytic sets are analytic). Let then $X = (\Omega, \mathcal{O}_\Omega)$, where $\Omega \subset \mathbb{C}^n$ open, and let A be an analytic subset of X. Then, by Theorem 9.9, the sheaf of ideals $\mathfrak{J}_A$ is $\mathcal{O}_\Omega$-coherent. For a point $\xi \in A$, let U be an open neighbourhood of ξ in Ω, and let $h_1, \dots, h_s \in \mathcal{O}_\Omega(U)$ be such that the germs $h_{1,x}, \dots, h_{s,x}$ generate $\mathfrak{J}_{A,x}$ for every $x \in U$.
Suppose first that A is irreducible at ξ, of dimension k; then $\mathrm{reg}A \cap U$ is a k-dimensional submanifold of U (possibly after shrinking U). Let $D_{\lambda\mu}(x)$ be the determinant with rows and columns

$$\lambda = (\lambda_1, \dots, \lambda_{n-k}), \quad \mu = (\mu_1, \dots, \mu_{n-k}),$$

respectively, from the matrix $\left[\dfrac{\partial h_i}{\partial x_j}(x)\right]_{\substack{i=1,\dots,s\\ j=1,\dots,n}}$, where $x \in U$. Then each $D_{\lambda\mu}$ is holomorphic on U, and we claim that

$$x \in \mathrm{sng}A \cap U \quad \Leftrightarrow \quad x \in \bigcap_{\lambda,\mu} D_{\lambda\mu}^{-1}(0).$$

First, suppose $x \in (A \cap U) \setminus \mathrm{sng}A$. Then x lies on a k-dimensional manifold $\mathrm{reg}A \cap U$, hence there are a small open neighbourhood $V \subset U$ of x and holomorphic $f_1, \dots, f_{n-k} \in \mathcal{O}(V)$ such that $\mathrm{rk}\left[\dfrac{\partial f_i}{\partial x_j}(x)\right] = n-k$, and all f_i vanish on $A \cap V = \mathrm{reg}A \cap V$. But then $f_{1,x}, \dots, f_{n-k,x}$ are generated by the $h_{1,x}, \dots, h_{s,x}$, and hence some $D_{\lambda\mu}$ must be non-zero at x (for otherwise $\mathrm{rk}\,[\partial f_i/\partial x_j(x)] < n-k$).

Suppose now that $x \in A \cap U$ and $D_{\lambda\mu}(x) \neq 0$ for some λ, μ. Then the differentials $dh_{\lambda_1}(x), \dots, dh_{\lambda_{n-k}}(x)$ are linearly independent, and hence the set $Z = \bigcap_{i=1}^{n-k} h_{\lambda_i}^{-1}(0)$ is a k-dimensional submanifold of a neighbourhood V of x in U. But $A \cap V \subset Z$ is a k-dimensional analytic subset of Z, hence $A \cap V = Z$, by irreducibility of Z_x (Prop. 7.6). Therefore $A \cap V \subset \mathrm{reg}A$, and thus $x \in A \setminus \mathrm{sng}A$, as required.

Now, for an arbitrary A, let $A_\xi = A_\xi^1 \cup \dots \cup A_\xi^q$ be the decomposition into irreducible germs, and let U be an open neighbourhood of ξ for which

$$A \cap U = A^1 \cup \dots \cup A^q,$$

where the A^j are representatives in U of the respective A_ξ^j. Then the result follows from the first part of the proof and the formula (7.2). □

9.4 Nilradical and reduction

Definition 9.17. Given a complex analytic space X, the *nilradical* of X is the sheaf $\mathcal{N}_X$ of ideals associated to the presheaf

$$\left\{f \in \mathcal{O}_X(U) : f^k = 0 \text{ for some } k \in \mathbb{N}\right\}_{U \in \mathfrak{Top}|X|}.$$

Then, for every $\xi \in |X|$, we have $\mathcal{N}_{X,\xi} = \mathrm{rad}(0)$.
Let C_X denote the sheaf of continuous complex-valued functions on $|X|$. For every open $U \subset |X|$, there is a canonical ring homomorphism

$$\mathcal{O}_X(U) \ni f \mapsto \tilde{f} \in C_X(U),$$

where $\tilde{f}(\xi) = f(\xi)$ ($\in \mathcal{O}_{X,\xi}/\mathfrak{m}_{X,\xi} \cong \mathbb{C}$) for $\xi \in U$, and this defines a sheaf morphism $\mathcal{O}_X \to C_X$.

Theorem 9.18. *Let $(X, \mathcal{O}_X)$ be a complex analytic space. Then*

(i) the nilradical $\mathcal{N}_X$ is a coherent $\mathcal{O}_X$-ideal

(ii) $\mathcal{N}_X = \ker\{\mathcal{O}_X \to C_X\}$.

Proof. The question being local, we may assume that X is a local model $(A, (\mathcal{O}_\Omega/\mathcal{I})|_A)$. Then, by Nullstellensatz,

$$\mathcal{N}_X = ((\mathrm{rad}\mathcal{I})/\mathcal{I})|_A = (\mathfrak{J}_A/\mathcal{I})|_A\,,$$

because this is so for stalks at every point of Ω. Hence Cartan's Coherence together with Lemma 8.29 imply (i). To prove (ii), it suffices to show that $\mathcal{N}_{X,\xi} = \ker(\mathcal{O}_{X,\xi} \to C_{X,\xi})$ for all $\xi \in X$. Take $\xi \in |X|$, and $f_\xi = F_\xi + \mathcal{I}_\xi \in (\mathcal{O}_\Omega/\mathcal{I})_\xi$, where $F_\xi \in \mathcal{O}_{\Omega,\xi}$. Then

$$(\tilde{f})_\xi = 0 \ \Leftrightarrow\ F_\xi \in \mathfrak{J}_{A,\xi} \ \Leftrightarrow\ f_\xi \in \mathfrak{J}_{A,\xi}/\mathcal{I}_\xi = \mathcal{N}_{X,\xi}\,.$$

□

We note another simple consequence of Nullstellensatz and coherence:

Proposition 9.19. *Let X be a complex analytic space, and let $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$ be coherent ideals. If*

$$\mathrm{supp}(\mathcal{O}_X/\mathcal{I}) \subset \mathrm{supp}(\mathcal{O}_X/\mathcal{J})\,,$$

then, for every relatively compact open $U \subset |X|$, there is an integer $k \in \mathbb{N}$ such that $\mathcal{J}^k|_U \subset \mathcal{I}|_U$. In particular, for every relatively compact open $U \subset |X|$, there exists $k \in \mathbb{N}$ such that

$$\mathcal{N}^k|_U = (0)\,.$$

Proof. Since $\mathrm{supp}(\mathcal{O}_X/\mathcal{I}) = \mathrm{supp}(\mathcal{O}_X/\mathrm{rad}\mathcal{I})$, and (by Nullstellensatz)

$$\mathrm{supp}(\mathcal{O}_X/\mathrm{rad}\mathcal{I}) \subset \mathrm{supp}(\mathcal{O}_X/\mathrm{rad}\mathcal{J}) \ \Leftrightarrow\ \mathrm{rad}\mathcal{J} \subset \mathrm{rad}\mathcal{I}\,,$$

it suffices to show that, for every coherent $\mathcal{I} \subset \mathcal{O}_X$ and relatively compact U, $(\mathrm{rad}\mathcal{I})^k|_U \subset \mathcal{I}|_U$ for some $k \in \mathbb{N}$. By noetherianity of $\mathcal{O}_{X,\xi}$, for every $\xi \in U$, we have $((\mathrm{rad}\mathcal{I})_\xi)^{k_\xi} \subset \mathcal{I}_\xi$ for some $k_\xi \in \mathbb{N}$, hence $(\mathrm{rad}\mathcal{I})^{k_\xi}|_{V^\xi} \subset \mathcal{I}|_{V^\xi}$ on some open $V^\xi \subset U$ around ξ (Corollary 8.31). Choosing $k = \max k_\xi$ over the finitely many points such that the corresponding V^ξ cover U, we get $(\mathrm{rad}\mathcal{I})^k|_U \subset \mathcal{I}|_U$, as required.

The second assertion now follows from equality $\mathrm{supp}(\mathcal{O}_X/\mathcal{N}_X) = \mathrm{supp}(\mathcal{O}_X/(0))$. □

Let $X = (|X|, \mathcal{O}_X)$ be a complex analytic space, and consider the ideal sheaf $\mathcal{N}_X$ on X. As $\mathcal{O}_X/\mathfrak{m}_{X,\xi} \cong \mathbb{C}$, for every $\xi \in |X|$, then $\mathcal{N}_{X,\xi} \neq \mathcal{O}_{X,\xi}$, hence, in particular, $\mathrm{supp}(\mathcal{O}_X/\mathcal{N}_X) = |X|$.

Definition 9.20. The closed complex analytic subspace of X

$$X_{\mathrm{red}} \ = (|X|, \mathcal{O}_X/\mathcal{N}_X)$$

is called the *reduction* of X. The space X is called *reduced* when $X = X_{\mathrm{red}}$ (i.e., when $\mathcal{N}_X = 0$).

We now want to show that every analytic mapping $\varphi : X \to Y$ of complex analytic spaces induces a canonical analytic morphism

$$\varphi_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$$

of their reductions. This will follow from a more general lemma below.

Lemma 9.21. *Let $\varphi : X \to Y$ be an analytic map, and assume closed complex analytic subspaces $X' \overset{\iota}{\hookrightarrow} X$ and $Y' \overset{\kappa}{\hookrightarrow} Y$ are defined by ideals $\mathcal{I} \subset \mathcal{O}_X$ and $\mathcal{J} \subset \mathcal{O}_Y$ respectively. Then there exists a (unique) analytic map $\varphi' : X' \to Y'$ satisfying $\varphi \circ \iota = \kappa \circ \varphi'$ if and only if $\varphi^*(\mathcal{J}) \subset |\varphi|_*\mathcal{I}$.*

Note that if $X' \subset X$ and $Y' \subset Y$ are open analytic subspaces, then the existence of the restriction $\varphi' : X' \to Y'$ is equivalent to the condition

$$|\varphi|(|X'|) \ \subset \ |Y'|$$

for the underlying topological spaces (**Exercise**). In the case of closed subspaces this condition is, of course, necessary, but in general, by no means sufficient.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc} & 0 & & 0 & \\ & \downarrow & & \downarrow & \\ & \mathcal{J} & & |\varphi|_*\mathcal{I} & \\ & \downarrow & & \downarrow & \\ & \mathcal{O}_Y & \xrightarrow{\varphi^*} & |\varphi|_*\mathcal{O}_X & \\ & \downarrow & & \downarrow & \\ |\kappa|_*\mathcal{O}_{Y'} = & \mathcal{O}_Y/\mathcal{J} & \overset{\alpha}{\dashrightarrow} & |\varphi|_*(\mathcal{O}_X/\mathcal{I}) & = |\varphi|_*|\iota|_*\mathcal{O}_{X'} \,. \\ & \downarrow & & \downarrow & \\ & 0 & & 0 & \end{array}$$

If φ' exists, we may define $\alpha := |\kappa|_*(\varphi')^*$, making the diagram commute. In particular,

$$|\varphi|_*\iota^* \circ \varphi^*(\mathcal{J}) = \alpha \circ \kappa^*(\mathcal{J}) = \alpha(0) = 0$$

implies $\varphi^*\mathcal{J} \subset \ker |\varphi|_*\iota^* = |\varphi|_*\mathcal{I}$.

Conversely, assume $\varphi^*\mathcal{J} \subset |\varphi|_*\mathcal{I}$. We first show that $|\varphi|(|X'|) \ \subset \ |Y'|$. Indeed, if $\xi \in \mathrm{supp}\mathcal{O}_X/\mathcal{I}$, that is, $\mathcal{I}_\xi \subset \mathfrak{m}_{X,\xi}$, then $\mathcal{J}_{|\varphi|.\xi} \subset \mathfrak{m}_{Y,|\varphi|.\xi}$, so $|\varphi|.\xi \in \mathrm{supp}\mathcal{O}_Y/\mathcal{J}$. We may thus define $|\varphi'|$ as the restriction of $|\varphi|$ to $|X'|$; i.e.,

$$|\varphi'| = |\kappa|^{-1} \circ |\varphi| \circ |\iota| \,.$$

Our assumption implies the existence of α making the diagram commute, and hence we may define

$$(\varphi')^* := |\kappa|^{-1}\alpha : \mathcal{O}_{Y'} = |\kappa|^{-1}(\mathcal{O}_Y/\mathcal{J}) \ \to \ |\kappa|^{-1}(|\varphi|_*(\mathcal{O}_X/\mathcal{I}))\,.$$

It remains to show that $|\kappa|^{-1}(|\varphi|_*(\mathcal{O}_X/\mathcal{I})) = |\varphi'|_*\mathcal{O}_{X'}$. And, indeed, by commutativity of the diagram and the properties of the "extension by zero" morphisms $|\iota|_*$ and $|\kappa|_*$, we get

$$\begin{aligned} |\varphi'|_*\mathcal{O}_{X'} = |\kappa|^{-1}|\kappa|_*|\varphi'|_*\mathcal{O}_{X'} = |\kappa|^{-1}|\kappa|_*|\varphi'|_*|\iota|^{-1}(\mathcal{O}_X/\mathcal{I}) \\ = |\kappa|^{-1}|\varphi|_*|\iota|_*|\iota|^{-1}(\mathcal{O}_X/\mathcal{I}) = |\kappa|^{-1}|\varphi|_*(\mathcal{O}_X/\mathcal{I})\,. \end{aligned}$$

□

Corollary 9.22. *If $\varphi : X \to Y$ is an analytic map, then there is a uniquely determined analytic mapping*

$$\varphi_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$$

such that $\varphi \circ \iota = \kappa \circ \varphi_{\mathrm{red}}$, where $\iota : X_{\mathrm{red}} \hookrightarrow X$ and $\kappa : Y_{\mathrm{red}} \hookrightarrow Y$ are the canonical embeddings.

Proof. Clearly, $\varphi^*\mathcal{N}_Y \subset |\varphi|_*\mathcal{N}_X$, as locally, $(\varphi^*\mathcal{N}_Y)^k = \varphi^*(\mathcal{N}_Y^k) = 0$ for some $k \in \mathbb{N}$. □

Example 9.23. 1. Let $X = \mathbb{C}^2$ with the coordinate functions x, y, let $Y = \mathbb{C}$ with coordinate z, and put $\varphi : X \ni (x,y) \mapsto x+y \in Y$. Then $\varphi^*(z) = x+y$. Now, let X' be the closed subspace of X defined by $\mathcal{I} = (x^2, y^2)$, and ley Y' be the subspace of Y defined by $\mathcal{J} = (z^2)$. Since $\varphi^*(z^2) = (\varphi^*(z))^2 = x^2+2xy+y^2$, it follows that $\varphi^*\mathcal{J} \not\subset |\varphi|_*\mathcal{I}$, and hence there is no restriction $\varphi' : X' \to Y'$ of φ.

2. Let, in turn, X be the analytic subspace of $\mathbb{C}^2$ defined by $\mathcal{I} = (x^2, y^2)$, let Y be the subspace of $\mathbb{C}$ defined by $\mathcal{J} = (z^2)$, and let $\varphi : X \to Y$ be given by $|\varphi|(0) = 0$ and $\varphi^*(z) = x^2 + y^2$. Then $\mathcal{N}_X = \mathrm{rad}\mathcal{I} = (x,y)$, $\mathcal{N}_Y = \mathrm{rad}\mathcal{J} = (z)$, so that X_{red} is a simple point, with $\mathcal{O}_{X_{\mathrm{red}}} = (\mathcal{O}_{\mathbb{C}^2}/(x,y))|_{X_{\mathrm{red}}}$, Y_{red} is a simple point, with $\mathcal{O}_{Y_{\mathrm{red}}} = (\mathcal{O}_{\mathbb{C}}/(z))|_{Y_{\mathrm{red}}}$, and $\varphi^*_{\mathrm{red}} = \mathrm{id}_{\mathbb{C}}$. On the other hand, $\varphi^*(z^2) = (\varphi^*(z))^2 = x^4 + 2x^2y^2 + y^4$, so $\varphi^*\mathcal{J} \subset |\varphi|_*\mathcal{I}$, and thus $\varphi : X \to Y$ itself is a restriction of $\Phi : \mathbb{C}^2 \ni (x,y) \mapsto x^2 + y^2 \in \mathbb{C}$.

We will now show that the continuous mapping $|\varphi| : |X| \to |Y|$ of the underlying topological spaces is determined by the sheaf morphism φ^*. Conversely, the sheaf morphism φ^* component of an analytic map $\varphi : X \to Y$ is uniquely determined by $|\varphi|$, provided $X = X_{\mathrm{red}}$. We begin with a simple observation:

Lemma 9.24. *Let $(R, \mathfrak{m})$ be a local analytic $\mathbb{C}$-algebra, and let $r_1, \dots, r_n \in \mathfrak{m}$ be given. Then there exists a unique homomorphism $\Phi : \mathbb{C}\{z_1, \dots, z_n\} \to R$ of local analytic $\mathbb{C}$-algebras satisfying $\Phi(z_j) = r_j$, $j = 1, \dots, n$.*

Let $X = (|X|, \mathcal{O}_X)$ be a complex analytic space. For $n \in \mathbb{N}$, we will denote by $\mathrm{Hol}(X, \mathbb{C}^n)$ the set of analytic mappings $X \to \mathbb{C}^n$, and by $z_1, \dots, z_n \in \mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n)$ the coordinate functions on $\mathbb{C}^n$.

Proposition 9.25. *If X is a complex analytic space and $U \subset |X|$ an open subset, then the map*

$$\mathrm{Hol}(U, \mathbb{C}^n) \ni \varphi \ \mapsto \ (\varphi^* z_1, \dots \varphi^* z_n) \in (\mathcal{O}_X(U))^n$$

is bijective.

Proof. For the proof of injectivity, let $\varphi, \psi \in \mathrm{Hol}(X, \mathbb{C}^n)$ be such that $\varphi^* z_j = \psi^* z_j$ for $j = 1, \dots, n$. For a point $\xi \in |X|$, let $c^j_\xi \in \mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n)$ (resp. $d^j_\xi \in \mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n)$) be the constant section $z \mapsto z_j(|\varphi|(\xi)) \in \mathbb{C}$ (resp. $z \mapsto z_j(|\psi|(\xi)) \in \mathbb{C}$). Then

$$(z_j)_{|\varphi|(\xi)} - (c^j_\xi)_{|\varphi|(\xi)} \in \mathfrak{m}_{\mathbb{C}^n, |\varphi|(\xi)}\,, \quad \text{hence} \quad \varphi^*_\xi[(z_j)_{|\varphi|(\xi)}] - \varphi^*_\xi[(c^j_\xi)_{|\varphi|(\xi)}] \in \mathfrak{m}_{X,\xi}\,,$$

and consequently

$$(\varphi^* z_j)_\xi + \mathfrak{m}_{X,\xi} = \varphi^*_\xi[(z_j)_{|\varphi|(\xi)}] + \mathfrak{m}_{X,\xi} = z_j(|\varphi|(\xi))\,.$$

Similarly,

$$(z_j)_{|\psi|(\xi)} - (d^j_\xi)_{|\psi|(\xi)} \in \mathfrak{m}_{\mathbb{C}^n, |\psi|(\xi)}\,, \quad \text{hence} \quad \psi^*_\xi[(z_j)_{|\psi|(\xi)}] - \psi^*_\xi[(d^j_\xi)_{|\psi|(\xi)}] \in \mathfrak{m}_{X,\xi}\,,$$

and consequently

$$(\psi^* z_j)_\xi + \mathfrak{m}_{X,\xi} = \psi^*_\xi[(z_j)_{|\psi|(\xi)}] + \mathfrak{m}_{X,\xi} = z_j(|\psi|(\xi))\,.$$

Thus

$$z_j(|\varphi|(\xi)) = (\varphi^* z_j)_\xi + \mathfrak{m}_{X,\xi} = (\psi^* z_j)_\xi + \mathfrak{m}_{X,\xi} = z_j(|\psi|(\xi))\,, \quad j = 1, \dots, n\,,$$

hence $|\varphi|(\xi) = |\psi|(\xi)$. As ξ was arbitrary, it follows that $|\varphi| = |\psi|$.
On the other hand, for $j = 1, \dots, n$,

$$\varphi^*_\xi(z_j) = (\varphi^* z_j)_\xi = (\psi^* z_j)_\xi = \psi^*_\xi(z_j)\,,$$

hence, by Lemma 9.24, $\varphi^*_\xi = \psi^*_\xi$. As ξ was arbitrary, we get $\varphi^* = \psi^*$.

Suppose now that $(f_1, \dots, f_n) \in (\mathcal{O}_X(U))^n$ are given. We may assume that U is a local model in $\Omega \subset \mathbb{C}^m$, generated by a coherent ideal $\mathcal{I}$, and the $f_j \in \mathcal{O}_X(U) = (\mathcal{O}_\Omega/\mathcal{I})|_U$ are induced by some $F_1, \dots, F_n \in \mathcal{O}_\Omega(\Omega)$ (after shrinking Ω if necessary). Consider the analytic map

$$F = (F_1, \dots, F_n) : \Omega \to \mathbb{C}^n ,$$

and its restriction $\varphi = F|_U : U \to \mathbb{C}^n$. Then $\varphi^* z_j = f_j$ for $j = 1, \dots, n$, which completes the proof. □

Corollary 9.26. *Let $X = (|X|, \mathcal{O}_X)$ be a complex analytic space. The following conditions are equivalent:*

(i) X is reduced.

(ii) For any open $U \subset |X|$, complex analytic space Y, and analytic maps $\varphi, \psi : U \to Y$, the equality $|\varphi| = |\psi|$ implies $\varphi = \psi$.

Proof. $(i) \Rightarrow (ii)$: The question being local, we may assume that $Y = \mathbb{C}^n$. Then, by the above Proposition 9.25, $\varphi : U \to \mathbb{C}^n$ (resp. $\psi : U \to \mathbb{C}^n$) is uniquely determined by the n-tuple of global sections $(\varphi^* z_1, \dots, \varphi^* z_n) \in (\mathcal{O}_X(U))^n$ (resp. $(\psi^* z_1, \dots, \psi^* z_n) \in (\mathcal{O}_X(U))^n$), which, in turn, can be identified with their evaluations, that is, images under the now injective (Theorem 9.18) sheaf morphism $\mathcal{O}_X \to C_X$, namely $(z_1 \circ |\varphi|, \dots, z_n \circ |\varphi|)$ and $(z_1 \circ |\psi|, \dots, z_n \circ |\psi|)$ respectively; the latter being equal, by assumption.
$(ii) \Rightarrow (i)$: If X is not reduced, there exists an open $U \subset |X|$, and a nilpotent $f \in \mathcal{O}_X(U)$, hence an analytic map $f : U \to \mathbb{C}$ which is not zero whilst $|f| = 0$. □

Example 9.27. The following are two distinct morphisms of a double point to itself: Define φ^* to send $1 \mapsto 1$ and $z \mapsto z$, and ψ^* to send $1 \mapsto 1$ and $z \mapsto z^2$. Then $(|\varphi|, \varphi^*) \neq (|\psi|, \psi^*)$.

9.5 Germs of complex analytic spaces - duality

Definition 9.28. If $X = (|X|, \mathcal{O}_X)$ is a complex analytic space, and $\xi \in |X|$, the pair (X, ξ) is called a *germ of a complex analytic space* X at ξ. A *morphism of germs* $(X, \xi) \to (Y, \eta)$ is a germ of an analytic map $X \to Y$. For an open $U \subset |X|$, a point $\xi \in U$, and analytic $\varphi : U \to Y$ with $\varphi(\xi) = \eta$, we denote the induced germ by $\varphi_\xi : (X, \xi) \to (Y, \eta)$.

The assignment

$$(X, \xi) \mapsto \mathcal{O}_{X,\xi}$$

becomes a contravariant functor from the category of germs of complex analytic spaces to the category of local analytic $\mathbb{C}$-algebras in the following way: If a morphism $\varphi_\xi : (X, \xi) \to (Y, \eta)$ is represented by an analytic map $\varphi : U \to Y$, where U is an open neighbourhood of ξ in $|X|$, then

$$\varphi^*_\xi : \mathcal{O}_{Y,\eta} \to \mathcal{O}_{X,\xi}$$

is the homomorphism of local rings induced by φ. The following result will be fundamental for our study of the local geometry of analytic mappings.

Proposition 9.29. *The functor defined above is an antiequivalence. That is:*

(i) If $\varphi_\xi, \psi_\xi : (X, \xi) \to (Y, \eta)$ are two morphisms of germs, then $\varphi^*_\xi = \psi^*_\xi$ implies $\varphi_\xi = \psi_\xi$.

(ii) If R is any local analytic $\mathbb{C}$-algebra, then there exists a germ (X, ξ) and an isomorphism $\mathcal{O}_{X,\xi} \to R$ of local analytic $\mathbb{C}$-algebras.

(iii) If $\theta : R \to S$ is a homomorphism of local analytic $\mathbb{C}$-algebras, then there is a morphism $\varphi_\xi : (X, \xi) \to (Y, \eta)$ of germs of analytic spaces, such that $\theta = \varphi_\xi^*$.

Proof. (*i*): Take an open neighbourhood $U \subset |X|$ of ξ, and analytic maps $\varphi, \psi : U \to Y$ representing φ_ξ and ψ_ξ respectively. The question being local, we may assume that Y is a local model, and further, that $Y = \mathbb{C}^n$ with $\eta = 0$. By Proposition 9.25, we may identify

$$\varphi = (\varphi^* z_1, \dots, \varphi^* z_n) \in (\mathcal{O}_X(U))^n \quad \text{and} \quad \psi = (\psi^* z_1, \dots, \psi^* z_n) \in (\mathcal{O}_X(U))^n . \tag{9.3}$$

The equality $\varphi_\xi^* = \psi_\xi^* : \mathbb{C}\{z_1, \dots, z_n\} \to \mathcal{O}_{X,\xi}$ implies, for $j = 1, \dots, n$,

$$(\varphi^* z_j)_\xi = \varphi_\xi^*(z_j) = \psi_\xi^*(z_j) = (\psi^* z_j)_\xi \,,$$

hence

$$\varphi^* z_1 = \psi^* z_1, \dots, \varphi^* z_n = \psi^* z_n \,,$$

after shrinking U if necessary. By (9.3), $\varphi_\xi = \psi_\xi$.

(*ii*): Write $R = \mathbb{C}\{w_1, \dots, w_m\}/\mathfrak{a}$, where $\mathfrak{a} = (f_1, \dots, f_r)$. By coherence, there is an open neighbourhood $W \subset \mathbb{C}^m$ of $\xi = 0 \in \mathbb{C}^m$, and holomorphic $F_1, \dots, F_r \in \mathcal{O}_W(W)$ representing the $f_1, \dots, f_r$. Denote by $\mathcal{I} \subset \mathcal{O}_W$ the sheaf of ideals generated by $F_1, \dots, F_r$ (Corollary 8.33), and by $X \hookrightarrow W$ the corresponding closed analytic subspace. Then

$$\mathcal{O}_{X,\xi} = (\mathcal{O}_X)_\xi \cong (\mathcal{O}_W/\mathcal{I})_\xi = (\mathcal{O}_W)_\xi/\mathcal{I}_\xi = \mathbb{C}\{w_1, \dots, w_m\}/\mathfrak{a} = R \,.$$

(*iii*): Let $\theta : R \to S$ be given. By (*ii*), we may write $\theta : \mathcal{O}_{Y,\eta} \to \mathcal{O}_{X,\xi}$, for some analytic spaces X and Y. We may assume that $Y \hookrightarrow W \subset \mathbb{C}^n$ is a local model, with $\eta = 0$, and consider the commutative diagram

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
 & & \mathfrak{a} \\
 & & \downarrow \\
 & & \mathbb{C}\{w_1, \dots, w_n\} \\
 & \overset{\Theta}{\swarrow} & \downarrow \\
\mathcal{O}_{X,\xi} & \overset{\theta}{\longleftarrow} & \mathcal{O}_{Y,\eta} \\
 & & \downarrow \\
 & & 0
\end{array}$$

where $\mathfrak{a} \in \mathbb{C}\{w_1, \dots, w_n\}$ is the ideal defining $\mathcal{O}_{Y,\eta}$, and Θ (the diagonal arrow in the traingle of the diagram) is defined by means of Lemma 9.24. By Proposition 9.25, there is an open neighbourhood $U \subset |X|$ of ξ, and an analytic map $\Phi : U \to \mathbb{C}^n$, such that $|\Phi|(\xi) = 0$ and $\Phi_\xi^* = \Theta$. Since $\Phi_\xi^*(\mathfrak{a}) = \Theta(\mathfrak{a}) = 0$, we may apply Lemma 9.21, and (after shrinking U, perhaps) obtain an analytic map $\varphi : U \to Y$ satisfying $\varphi_\xi^* = \theta$. □

The duality stated in Proposition 9.29 allows to derive local geometric properties of complex analytic spaces and analytic mappings from algebraic considerations on the local analytic $\mathbb{C}$-algebras. We complete this section with two immediate consequences.

Corollary 9.30. *For a complex analytic space $X = (|X|, \mathcal{O}_X)$ and a point $\xi \in |X|$, the following conditions are equivalent:*

(i) X is smooth at ξ.

(ii) The local ring $\mathcal{O}_{X,\xi}$ is regular.

Proof. $(i) \Rightarrow (ii)$ is clear. If, in turn, the local ring $\mathcal{O}_{X,\xi}$ is regular, then there is an isomorphism

$$\mathbb{C}\{z_1, \dots, z_n\} \to \mathcal{O}_{X,\xi}$$

for some $n \in \mathbb{N}$, which by Proposition 9.29(iii), yields an isomorphism of complex analytic spaces $\varphi : U \to W$, where U is an open neighbourhood of ξ in X, and W an open neighbourhood of zero in $\mathbb{C}^n$. □

Corollary 9.31. *Let $V \subset \mathbb{C}^m$ and $W \subset \mathbb{C}^n$ be open, let $X \overset{\iota}{\hookrightarrow} V$ and $Y \overset{\kappa}{\hookrightarrow} W$ be closed analytic subspaces, and let $\varphi : X \to Y$ be an analytic mapping. Then, for every $\xi \in X$, there exists an open neighbourhood V' of ξ in V, and an analytic map $\Phi : V' \to W$, such that $\Phi \circ (\iota|_{X \cap V'}) = \kappa \circ (\varphi|_{X \cap V'})$.*

Proof. We may assume that $\xi = 0 \in \mathbb{C}^m$ and $\varphi(\xi) = 0 \in \mathbb{C}^n$, and consider the diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathfrak{a} & & \mathfrak{b} \\
\downarrow & & \downarrow \\
\mathbb{C}\{w_1, \dots, w_m\} & \overset{\theta}{\longleftarrow\!-\!-} & \mathbb{C}\{z_1, \dots, z_n\} \\
\rho \downarrow & \overset{\alpha}{\swarrow} & \downarrow \sigma \\
\mathcal{O}_{X,0} & \overset{\varphi_0^*}{\longleftarrow} & \mathcal{O}_{Y,0} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

where ρ, σ are the canonical homomorphisms, and $\alpha := \varphi_0^* \circ \sigma$. Define $f_j = \alpha(z_j) \in \mathcal{O}_{X,0}$ for $j = 1, \dots, n$, and choose $F_1, \dots, F_n \in \mathbb{C}\{w_1, \dots, w_m\}$ such that $f_j = \rho(F_j)$. By Lemma 9.24, there is a homomorphism

$$\theta : \mathbb{C}\{z_1, \dots, z_n\} \to \mathbb{C}\{w_1, \dots, w_m\}$$

with $\theta(z_j) = F_j$. Then, by Proposition 9.29(iii), there are an open neighbourhood $V' \subset V$ of $0 \in \mathbb{C}^m$ and an analytic $\Phi : V' \to \mathbb{C}^n$, such that $\Phi_0^* = \theta$. Since θ induces φ_0^*, it follows that φ is a restriction of Φ. □

Remark 9.32. In the second part of these notes, we will use notation X_ξ for a germ (X, ξ) of an analytic space X at a point $\xi \in |X|$ (i.e., the same notation as was used for an analytic set germ). It will be always clear from the context whether we mean X_ξ as a germ of an analytic space or as an analytic set germ. In any case, if X is reduced at ξ, then there is no difference.