BRAUER GROUPS OF ALGEBRAIC STACKS AND GIT-QUOTIENTS:I

AJNEET DHILLON, JAYA NN IYER, AND ROY JOSHUA

Abstract. In this paper we consider the Brauer groups of algebraic stacks and GIT quotients: the only algebraic stacks we consider in this paper are quotient stacks [X/G] with certain restrictions on X and G, and defined over a Dedekind domain, a discrete valuation ring, or a field. We discuss the calculation of the Brauer groups of various examples. This will be continued in a sequel, where we will discuss more examples, especially the Brauer groups of various moduli stacks of principal G-bundles on a smooth projective curve X, associated to a reductive group G as well the Brauer groups of their GIT quotients.

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1. Introduction and the Main Results

The paper originated in an effort by the authors to study the Brauer groups quotient stacks and of GIT quotients associated to actions of reductive groups. We began by assuming the base scheme is a separably closed field, then soon extended our framework to the case where it is any field. While working on various examples, we realized that as several of the algebraic stacks one encounters often are defined over the ring of integers or Dedekind domains, it is preferable to adopt a more general framework as follows. This enables us to consider cohomological invariants of algebraic stacks defined over arbitrary Dedekind domains. The only cohomological invariants we consider will be the ℓ-primary torsion part of the Brauer group, where ℓ is a prime different from the residue characteristics.

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Let B denote a regular Noetherian scheme of dimension at most 1: B will serve as the *base scheme*. We also consider two basic situations here:

1.1. **Basic hypotheses.**

(i) $B = \operatorname{Spec} R$, where R is a Dedekind domain, or a DVR (for example, the ring of integers $\mathbb{Z}$ or its localization at a prime p), *or*

(ii) B is a smooth scheme of dimension at most 1 over a field k.

Observe that in any case B is a regular integral Noetherian domain of dimension at most 1. Let ℓ denote a fixed prime invertible in $\mathcal{O}_B$ and let X denote a scheme of finite type over B. Then one begins with the *Kummer sequence*

$$1 \to \mu_{\ell^n}(1) \to \mathbb{G}_m \overset{\ell^n}{\to} \mathbb{G}_m \to 1, \tag{1.1}$$

which holds on the (small) étale site X_{et} of X, whenever ℓ is invertible in $\mathcal{O}_B$. (See [Gr, section 3] or [Mi, p. 66].) Taking étale cohomology, we obtain corresponding long-exact sequence:

$$\to H^1_{et}(X, \mathbb{G}_m) \overset{\ell^n}{\to} H^1_{et}(X, \mathbb{G}_m) \to H^2_{et}(X, \mu_{\ell^n}(1)) \to H^2_{et}(X, \mathbb{G}_m) \to H^2_{et}(X, \mathbb{G}_m) \to \cdots, \tag{1.2}$$

which holds on the étale site when ℓ is invertible in $\mathcal{O}_B$.

Definition 1.1. *The cohomological Brauer group* $\mathrm{Br}(X)$ *is the torsion subgroup of the cohomology group* $H^2_{et}(X, \mathbb{G}_m)$. *In other words,* $\mathrm{Br}(X) = H^2_{et}(X, \mathbb{G}_m)_{tors}$. [1]

Next assume X is smooth over the base scheme B. Then, by Hilbert's Theorem 90, we obtain the isomorphisms:

$$\mathrm{Pic}(X) \cong \mathrm{CH}^1(X) \cong H^1_{et}(X, \mathbb{G}_m) \cong H^{2,1}_M(X, \mathbb{Z}), \tag{1.3}$$

where $H^{2,1}_M(X, \mathbb{Z})$ denotes motivic cohomology (in degree 2 and weight 1) whose definition for smooth schemes of finite type over B is worked out in [Geis], and we recall this in the Appendix. Then one also obtains the short-exact sequence:

$$0 \to \mathrm{Pic}(X)/\ell^n \cong \mathrm{NS}(X)/\ell^n \to H^2_{et}(X, \mu_{\ell^n}(1)) \to \mathrm{Br}(X)_{\ell^n} \to 0, \tag{1.4}$$

where the map $\mathrm{Pic}(X)/\ell^n = H^{2,1}_M(X, \mathbb{Z}/\ell^n) \to H^2_{et}(X, \mu_{\ell^n}(1))$ is the cycle map, and therefore, $\mathrm{Br}(X)_{\ell^n}$ identifies with the cokernel of the cycle map. Thus it follows that for smooth schemes X over B, $\mathrm{Br}(X)_{\ell^n}$ is trivial if and only if the above cycle map is surjective: our approach to the Brauer group adopted in this paper is to consider the above cycle map from motivic cohomology to etale cohomology, and involves a combination of motivic and étale cohomology techniques. Moreover, in view of this, *we will always restrict to smooth schemes of finite type over the given base scheme* B.

Let G denote a not-necessarily connected smooth affine group scheme, of finite type over B, and acting on the given scheme X. Next we recall the framework of Borel-style equivariant étale cohomology, and Borel-style equivariant motivic cohomology. For this we form an ind-scheme $\{EG^{gm,m} \underset{G}{\times} X | m\}$ and then take its étale cohomology, and also its motivic cohomology when X is also assumed to be smooth. One may consult [Tot99], [MV99], and also section 3 of this paper for more details. Here $BG^{gm,m}$ is a finite dimensional approximation to the classifying space of the affine group scheme G, and $EG^{gm,m}$ denotes the universal

[1] If X is a regular integral Noetherian scheme, one may observe that $H^2_{et}(X, \mathbb{G}_m)_{tors} = H^2_{et}(X, \mathbb{G}_m)$: see, for example, [CTS, Lemma 3.5.3]

principal G-bundle over $\mathrm{BG^{gm,m}}$. In the terminology of Definition 3.1, $\mathrm{EG^{gm,m} = U_m}$ and $\mathrm{BG^{gm,m} = U_m/G}$. We also assume that such a $\mathrm{BG^{gm,m}}$ exists, for every $m \geq 0$, as a quasi-projective scheme over the given base B. There are standard arguments to prove that that the cohomology of the ind-schemes $\{\mathrm{BG^{gm,m}|m \geq 0}\}$, $\{\mathrm{EG^{gm,m} \times_G X|m \geq 0}\}$ are independent of the choice of the admissible gadgets $\{\mathrm{U_m|m \geq 0}\}$ that enter into their definition: see, for example, Proposition 3.8.

Let ℓ denote a fixed prime invertible in $\mathcal{O}_{\mathrm{B}}$. Then we let $\mathrm{H^{*,\bullet}_{G,M}(X, \mathbb{Z}/\ell^n)}$ denote the motivic cohomology of $\{\mathrm{EG^{gm,m}\underset{G}{\times}X|m}\}$ defined as the homotopy inverse limit of the motivic cohomology of the finite dimensional approximations $\mathrm{EG^{gm,m}\underset{G}{\times}X}$, that is, defined by the usual Milnor exact sequence relating lim^1 and lim of the motivic hypercohomology of the above finite dimensional approximations. (When $* = 2i$ and $\bullet = i$, for a non-negative integer i, these identify with the usual (equivariant) Chow groups.) $\mathrm{H^*_{G,et}(X, \mu_{\ell^n}(\bullet))}$ is defined similarly and will be often denoted $\mathrm{H^{*,\bullet}_{G,et}(X, \mu_{\ell^n})}$.

Recall that for each fixed integer $i \geq 0$, one obtains the isomorphisms (for m chosen, depending on i):

$$\mathrm{H^{2i,i}_{G,M}(X, \mathbb{Z}/\ell^n) \cong H^{2i,i}_M(EG^{gm,m}\underset{G}{\times}X, Z/\ell^n), m >> 0 \text{ and X smooth, and}}$$

$$\mathrm{H^{2i,i}_{G,et}(X, \mu_{\ell^n}) \cong H^{2i}_{et}(EG^{gm,m}\underset{G}{\times}X, \mu_{\ell^n}(i)), m >> 0.}$$

These show that one may define the G-*equivariant Brauer group* of a G-scheme X as follows:

Definition 1.2. $\mathrm{Br_G(X) = H^2_{et}(EG^{gm,m}\underset{G}{\times}X, \mathbb{G}_m)_{tors}}$, *for $m >> 0$, where the subscript tors denotes the torsion subgroup.* [2]

Moreover, we obtain from the Kummer-sequence the short-exact sequence:

$$(1.5)\qquad \mathrm{0 \to Pic(EG^{gm,m}\underset{G}{\times}X)/\ell^n \to H^2_{et}(EG^{gm,m}\underset{G}{\times}X, \mu_{\ell^n}(1)) \to Br(EG^{gm,m}\underset{G}{\times}X)_{\ell^n} = Br_G(X)_{\ell^n} \to 0} \text{ and}$$

where

$\mathrm{Pic(EG^{gm,m}\underset{G}{\times}X)/\ell^n = coker(Pic(EG^{gm,m}\underset{G}{\times}X)\overset{\ell^n}{\to}Pic(EG^{gm,m}\underset{G}{\times}X))}$,
$\mathrm{Br_G(X)_{\ell^n}}$ = the ℓ^n-torsion part of $\mathrm{Br_G(X)}$.

The comparison theorem [J20, Theorem 1.6] shows that $\mathrm{H^*_{G,et}(X, \mu_{\ell^n}(\bullet))}$ identifies with $\mathrm{H^*_{smt}([X/G], \mu_{\ell^n})}$, which denotes the cohomology of the quotient stack [X/G] computed on the smooth site: see Proposition 3.6 below for further details. This motivates the the following definition.

Definition 1.3. *Given an Artin stack S of finite type over the base scheme* B*, we define its Brauer group to be $\mathrm{H^2_{smt}}(S, \mathbb{G}_m)_{tors}$, where $\mathrm{H^2_{smt}}(S, \mathbb{G}_m)$ denotes cohomology computed on the smooth site, and the subscript tors denotes its torsion subgroup. We denote this by* $\mathrm{Br}(S)$. *For a fixed prime $\ell \neq char(k)$, we let* $\mathrm{Br}(S)_{\ell^n}$ *denote the ℓ^n-torsion part of* $\mathrm{Br}(S)$.

Then our first result is the following, which shows the Brauer group of a quotient stack [X/G], so defined, identifies with the G-equivariant Brauer group defined in Definition 1.2.

Theorem 1.4. *Assume that* X *is a smooth scheme of finite type over the base scheme* B *satisfying one of the hypotheses 1.1, and provided with an action by the affine smooth group scheme* G. *Then, assuming the*

[2] Here we remind the reader that, despite the similarity in appearance, the above equivariant Brauer groups are quite different from what are called, *invariant Brauer groups*: see [Cao].

above terminology,

$$\mathrm{Br}([\mathrm{X}/\mathrm{G}])_{\ell^{\mathrm{n}}} \cong \mathrm{Br}_{\mathrm{G}}(\mathrm{X})_{\ell^{\mathrm{n}}}.$$

Therefore, $\mathrm{Br}_{\mathrm{G}}(\mathrm{X})_{\ell^{\mathrm{n}}}$ *is intrinsic to the quotient stack* $[\mathrm{X}/\mathrm{G}]$.

We derive a number of results based on the above theorem, a few of which are listed below.

Corollary 1.5. *Assume in addition to the hypotheses of Theorem 1.4 that* $\mathrm{Br}(\mathrm{X})_{\ell^{\mathrm{n}}} = 0$. *Then* $\mathrm{Br}([\mathrm{X}/\mathrm{G}])_{\ell^{\mathrm{n}}} \cong 0$ *as well in the following cases*

(i) G *is a split torus, or*
(ii) G *is a finite product of general linear groups.*

We define a *toric stack* $\mathcal{X}$ to be an algebraic stack of the following form: X denotes a smooth split toric variety over a field k, with dense split torus T and provided with a homomorphism $\phi : \mathrm{T}_0 = \mathbb{G}_{\mathrm{m}}^r \to \mathrm{T}$, for some $r > 0$. Then we require that $\mathcal{X} = [\mathrm{X}/\mathrm{T}_0]$.

Theorem 1.6. *(i) The scheme* $(\mathbb{A}^2 - \{0\})^r \times_{\mathrm{T}_0} \mathrm{X}$, *where* T_0 *acts on* X *through* ϕ *and it acts diagonally on* $(\mathbb{A}^2 - \{0\})^r \times \mathrm{X}$, *is a split toric variety over* k, *for the split torus* $\mathrm{T}_0 \times \mathrm{T}$.

(ii) Therefore, $\mathrm{Br}([\mathrm{X}/\mathrm{T}_0])_{\ell^{\mathrm{n}}}$ *is isomorphic to the* ℓ^n*-torsion part of the Brauer group of the toric variety* $(\mathbb{A}^2 - \{0\})^r \times_{\mathrm{T}_0} \mathrm{X}$.

As an application of Theorem 1.4 for stacks defined over Dedekind domains, we also obtain the following computations. Let $\mathrm{R} = \mathbb{Z}[1/6]$ and let $\mathcal{M}_{1,1,\mathrm{R}}$ denote the moduli stack of elliptic curves defined over R. Let $\mathrm{Y} = \mathrm{Spec}\, \mathrm{R}[\mathrm{g}_2, \mathrm{g}_3][1/\Delta] \subseteq \mathbb{A}^2_{\mathrm{R}}$, where $\Delta = g_2^3 - 27g_3^2$.

Theorem 1.7. *Let* $\mathrm{B} = \mathrm{Spec}\, \mathbb{Z}[1/6]$ *and let* Y *be as above. Let* $\mathbb{Z}[1/6]^*$ *denote the units in the ring* $\mathbb{Z}[1/6]$. *Then for* $\ell = 2$, *or* 3,

(i) $\mathrm{Br}(\mathrm{Y})_{\ell^{\mathrm{n}}} \cong (\mathbb{Z}/2\mathbb{Z})_{\ell^{\mathrm{n}}} \oplus (\mathbb{Q}/\mathbb{Z})_{\ell^{\mathrm{n}}} \oplus (\mathrm{coker}(\mathbb{Z}[1/6]^* \stackrel{\ell^{\mathrm{n}}}{\to} \mathbb{Z}[1/6]^*)$,
(ii) $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Y}, \mu_{\ell^{\mathrm{n}}}) \cong (\mathrm{coker}(\mathbb{Z}[1/6]^* \stackrel{\ell^{\mathrm{n}}}{\to} \mathbb{Z}[1/6]^*) \oplus (\mathrm{ker}(\mathbb{Z}[1/6]^* \stackrel{\ell^{\mathrm{n}}}{\to} \mathbb{Z}[1/6]^*)$, *and*
(iii) $\mathrm{Br}(\mathcal{M}_{1,1,\mathrm{R}})_{\ell^{\mathrm{n}}} = \mathrm{ker}(\mathrm{res} : \mathrm{Br}(\mathrm{Y})_{\ell^{\mathrm{n}}} \to \mathrm{H}^1_{\mathrm{et}}(\mathrm{Y}, \mu_{\ell^{\mathrm{n}}}))$, *where res denotes the residue map as* (2.10).

In fact, in Theorem 6.1, we have a more general calculation of the ℓ^n-torsion part of the Brauer group of the stack $\mathcal{M}_{1,1,\mathrm{R}}$, where R is any Dedekind domain.

Next we discuss the following application of the torsion index of linear algebraic groups: see [Tot05, section 1].

Theorem 1.8. *Assume the base field* k *is any field and that* ℓ *is a fixed prime different from char*(k). *If* H *is a connected linear algebraic group defined over* k *and whose torsion index is prime to* ℓ, *then* $\mathrm{Br}(\mathrm{BH})_{\ell^{\mathrm{n}}} = \mathrm{Br}(\mathrm{Spec}\, k)_{\ell^{\mathrm{n}}}$, *where* BH *denotes the classifying stack of* H, *that is* $[\mathrm{Spec}\, k/\mathrm{H}]$. *In particular, the following hold:*

(i) $\mathrm{Br}(\mathrm{BG})_{\ell^{\mathrm{n}}} = \mathrm{Br}(\mathrm{Spec}\, k)_{\ell^{\mathrm{n}}}$ *for any prime* ℓ *different from the characteristic of* k *if* $\mathrm{G} = \mathrm{GL}_{\mathrm{n}}$, $\mathrm{G} = \mathrm{SL}_{\mathrm{n}}$ *or* $\mathrm{G} = \mathrm{Sp}(2\mathrm{n})$, *for any* n.

(ii) $\mathrm{Br}(\mathrm{BG})_{\ell^{\mathrm{n}}} = \mathrm{Br}(\mathrm{Spec}\, k)_{\ell^{\mathrm{n}}}$ *for any prime* ℓ *different from the characteristic of* k *and* 2 *if* $\mathrm{G} = \mathrm{SO}(2\mathrm{n})$, $\mathrm{SO}(2n+1)$, *or* $\mathrm{Spin}(n)$, *for any* n.

(iii) $\mathrm{Br}(\mathrm{BG})_{\ell^n} = \mathrm{Br}(\mathrm{Spec}\, k)_{\ell^n}$ *for any simply-connected group* G*, if* ℓ *is different from the characteristic of* k *and also different from* $2, 3,$ *or* 5*.*

Next let X denote a smooth projective curve of genus g over a field k, provided with a k-rational point. Then one knows the isomorphism of stacks (see for example, [Wang, Proposition 4.2.5]):

$$\mathrm{Bun}_{1,d}(\mathrm{X}) \cong \mathrm{B}\mathbb{G}_{\mathrm{m}}^{\mathrm{gm}} \times \mathbf{P}\mathrm{ic}^{\mathrm{d}}(\mathrm{X}), \tag{1.6}$$

where $\mathrm{B}\mathbb{G}_m^{gm} = \lim_{n \to \infty} \mathrm{B}\mathbb{G}_m^{gm,n}$, $\mathrm{Bun}_{1,d}(\mathrm{X})$ denotes the moduli stack of line bundles of degree d on X and $\mathbf{Pic}^d(\mathrm{X})$ denotes the Picard scheme. In view of the above isomorphism of stacks, one may define the Brauer group of the stack $\mathrm{Bun}_{1,d}(\mathrm{X})$ to be the Brauer group of the stack $\mathrm{B}\mathbb{G}_m^{\mathrm{gm}} \times \mathbf{Pic}^d(\mathrm{X})$. Then, we obtain the following theorem.

Theorem 1.9. *Assume the base field* k *is separably closed. Then, assuming the above situation, we obtain the isomorphism:*

$$\mathrm{Br}(\mathrm{Bun}_{1,\mathrm{d}}(\mathrm{X}))_{\ell^n} \cong \mathrm{Br}(\mathbf{P}\mathrm{ic}^{\mathrm{d}}(\mathrm{X}))_{\ell^n} \cong \mathrm{Br}(\mathrm{Sym}^{\mathrm{d}}(\mathrm{X}))_{\ell^n}.$$

In particular, $\mathrm{Br}(\mathrm{Bun}_{1,\mathrm{d}}(\mathrm{X}))_{\ell^n} \cong 0$ *if* X *is rational.*

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2. Basic computational techniques

In this section we discuss a number of basic techniques that will help us compute the Brauer groups of various quotient stacks. We begin with the following result that computes the Brauer groups of schemes of the form $\mathrm{X} \times \mathbb{G}_{\mathrm{m}}^{\mathrm{q}}$, $q \geq 1$, over a base scheme B as in (1.1).

Lemma 2.1. *Consider the commutative diagram*

$$\begin{array}{ccccccccc}
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \longrightarrow & E' \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \eta} \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \longrightarrow & E
\end{array}$$

in an abelian category with exact rows. Then the following hold:

(i) *If* α *and* β *are isomorphisms and* δ *is a monomorphism, then the map* γ *is also a monomorphism.*

(ii) *If* α *is an epimorphism and* η *is a monomorphism, then*

$$kernel(\beta) \to kernel(\gamma) \to kernel(\delta) \to cokernel(\beta) \to cokernel(\gamma) \to cokernel(\delta)$$

is exact. In particular, if α *is an epimorphism,* η *is a monomorphism and both* β *and* δ *are isomorphisms, then so is* γ*.*

Proof. The proof of the first statement is a straight-forward diagram-chase, making strong use of the fact α and β are isomorphisms and δ is a monomorphism. Here is a an outline of a proof. Let $c' \epsilon C'$ be such that $\gamma(c') = 0$. Then $\delta(h'(c')) = h(\gamma(c')) = 0$. As δ is assumed to be a monomorphism, it follows $h'(c') = 0$. By the exactness of the top row, there exists a $b' \epsilon B'$ so that $g'(b') = c'$. Now $g(\beta(b')) = \gamma(g'(b')) = \gamma(c') = 0$, so that there exists an $a \epsilon A$ so that $f(a) = \beta(b')$. But as both α and β are isomorphism, there exists an $a' \epsilon A'$

so that $\alpha(a') = a$ and $f'(a') = b'$. But, then by the exactness of the top row, $c' = g'(b') = g'(f'(a')) = 0$. Thus γ must be a monomorphism, which proves the first statement. The second statement is a variant of the Snake Lemma: see, [Iver, Snake Lemma 1.6]. □

Proposition 2.2. *Let* X *denote a smooth scheme of finite type over* B*, and let* ℓ *denote a fixed prime different from the residue characteristics of* B*. Then the following hold:*

- *the localization sequence*

$$\mathrm{H}_{\mathrm{M}}^{*,\bullet}(\mathrm{X}\times\mathbb{A}^1,\mathbb{Z}/\ell^{\mathrm{n}}) \xrightarrow{j^*} \mathrm{H}_{\mathrm{M}}^{*,\bullet}(\mathrm{X}\times\mathbb{G}_{\mathrm{m}},\mathbb{Z}/\ell^{\mathrm{n}}) \longrightarrow \mathrm{H}_{\mathrm{M},\mathrm{X}\times\{0\}}^{*+1,\bullet}(\mathrm{X}\times\mathbb{A}^1,\mathbb{Z}/\ell^{\mathrm{n}}) \cong \mathrm{H}_{\mathrm{M}}^{*-1,\bullet-1}(\mathrm{X}\times\{0\},\mathbb{Z}/\ell^{\mathrm{n}})$$

  breaks up into short exact sequences, thereby providing the isomorphism:

$$\mathrm{H}_{\mathrm{M}}^{*,\bullet}(\mathrm{X}\times\mathbb{G}_{\mathrm{m}},\mathbb{Z}/\ell^{\mathrm{n}}) \cong \mathrm{H}_{\mathrm{M}}^{*,\bullet}(\mathrm{X},\mathbb{Z}/\ell^{\mathrm{n}}) \oplus \mathrm{H}_{\mathrm{M}}^{*-1,\bullet-1}(\mathrm{X},\mathbb{Z}/\ell^{\mathrm{n}}).$$

- *A corresponding result holds in étale cohomology with coefficients in* $\mu_{\ell^n}(i)$*.*

Proof. Let $p : \mathrm{X}\times\mathbb{A}^1 \to \mathrm{X}\times\{1\} \cong \mathrm{X}\times \mathrm{Spec}\, k$ denote the obvious projection and let $i : \mathrm{X}\times\mathrm{Spec}\, k \cong \mathrm{X}\times\{1\} \to \mathrm{X}\times\mathbb{G}_{\mathrm{m}}$ denote the corresponding closed immersion. Then it is clear that the induced map

$$p^* : \mathrm{H}_{\mathrm{M}}^{*,\bullet}(\mathrm{X}\times\{1\},\mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}_{\mathrm{M}}^{*,\bullet}(\mathrm{X}\times\mathbb{A}^1,\mathbb{Z}/\ell^{\mathrm{n}})$$

is an isomorphism and the composition of the maps

$$i^*\circ j^*\circ p^* : \mathrm{H}_{\mathrm{M}}^{*,\bullet}(\mathrm{X}\times\{1\},\mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}_{\mathrm{M}}^{*,\bullet}(\mathrm{X}\times\{1\},\mathbb{Z}/\ell^{\mathrm{n}})$$

is the identity, thereby proving that the map $j^* : \mathrm{H}_{\mathrm{M}}^{*,\bullet}(\mathrm{X}\times\mathbb{A}^1,\mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}_{\mathrm{M}}^{*,\bullet}(\mathrm{X}\times\mathbb{G}_{\mathrm{m}},\mathbb{Z}/\ell^{\mathrm{n}})$ is a split injection. This proves the required assertion. □

Corollary 2.3. *Let* X *denote a smooth scheme of finite type over* B*. Then the following hold:*

(i) *We obtain the isomorphism for any positive integer* q*:*

$$\mathrm{H}_{\mathrm{M}}^{2,1}(\mathrm{X}\times\mathbb{G}_{\mathrm{m}}^{\times \mathrm{q}},\mathbb{Z}/\ell^{\mathrm{n}}) \cong \mathrm{H}_{\mathrm{M}}^{2,1}(\mathrm{X},\mathbb{Z}/\ell^{\mathrm{n}}) \oplus (\oplus_{i=1}^{q}\mathrm{H}_{\mathrm{M}}^{1,0}(\mathrm{X},\mathbb{Z}/\ell^{\mathrm{n}})).$$

Moreover $\mathrm{H}_{\mathrm{M}}^{1,0}(\mathrm{X},\mathbb{Z}/\ell^{\mathrm{n}}) \cong 0$*, so that the last isomorphism becomes:*

$$\mathrm{H}_{\mathrm{M}}^{2,1}(\mathrm{X}\times\mathbb{G}_{\mathrm{m}}^{\times \mathrm{q}},\mathbb{Z}/\ell^{\mathrm{n}}) \cong \mathrm{H}_{\mathrm{M}}^{2,1}(\mathrm{X},\mathbb{Z}/\ell^{\mathrm{n}}).$$

(ii) *The corresponding result for étale cohomology with respect to* $\mu_{\ell^n}(i)$*, where* $\mu_{\ell^n}(0)$ *is identified with the constant sheaf* $\mathbb{Z}/\ell^n$*, is:*

$$\mathrm{H}_{\mathrm{et}}^{2}(\mathrm{X}\times\mathbb{G}_{\mathrm{m}}^{\times \mathrm{q}},\mu_{\ell^{\mathrm{n}}}(1)) \cong \mathrm{H}_{\mathrm{et}}^{2}(\mathrm{X},\mu_{\ell^{\mathrm{n}}}(1)) \oplus (\oplus_{i=1}^{q}\mathrm{H}_{\mathrm{et}}^{1}(\mathrm{X},\mu_{\ell^{\mathrm{n}}}(0))) \oplus (\oplus_{i=1}^{\binom{q}{2}}\mathrm{H}_{\mathrm{et}}^{0}(\mathrm{X},\mu_{\ell^{\mathrm{n}}}(-1))).$$

with the understanding that for $q=1$, $\binom{q}{2} = 0$.

(iii) *One also obtains the following isomorphisms for étale cohomology with respect to* $\mu_{\ell^n}(i)$ *where* $\mu_{\ell^n}(0)$ *is identified with the constant sheaf* $\mathbb{Z}/\ell^n$*:*

$$\mathrm{H}_{\mathrm{et}}^{1}(\mathrm{X}\times\mathbb{G}_{\mathrm{m}}^{\times \mathrm{q}},\mu_{\ell^{\mathrm{n}}}(1)) \cong \mathrm{H}_{\mathrm{et}}^{1}(\mathrm{X},\mu_{\ell^{\mathrm{n}}}(1)) \oplus (\oplus_{i=1}^{q}\mathrm{H}_{\mathrm{et}}^{0}(\mathrm{X},\mu_{\ell^{\mathrm{n}}}(0))).$$

(iv) *Consequently* $\mathrm{Br}(\mathrm{X}\times\mathbb{G}_{\mathrm{m}}^{\times \mathrm{q}})_{\ell^{\mathrm{n}}} \cong \mathrm{Br}(\mathrm{X})_{\ell^{\mathrm{n}}} \oplus (\oplus^{\mathrm{q}}\mathrm{H}_{\mathrm{et}}^{1}(\mathrm{X},\mu_{\ell^{\mathrm{n}}}(0))) \oplus (\oplus^{\binom{q}{2}}\mathrm{H}_{\mathrm{et}}^{0}(\mathrm{X},\mu_{\ell^{\mathrm{n}}}(-1)))$*, with the same understanding that* $\binom{q}{2} = 0$ *for* $q=1$.

Proof. The isomorphism in the first statement in (i) may be deduced from the last proposition by ascending induction on q. Here one may want to observe that the motivic complexes $\mathbb{Z}(j)$ are defined only for $j >= 0$, i.e., $\mathbb{Z}(j) = 0$, for $j < 0$. Next we consider the remaining statements in (i).

Observe that the term $\mathrm{H}^{1,0}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}})$

$$\mathrm{H}^{1,0}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}}) \cong \mathrm{H}^{3,1}_{\mathrm{M},\mathrm{X}\times\{0\}}(\mathrm{X}\times\mathbb{A}^1, \mathbb{Z}/\ell^{\mathrm{n}})$$
$$\cong \mathrm{H}^3_{\mathrm{Zar}}(\mathrm{X}\times\mathbb{A}^1, \mathrm{i}_*(\mathbb{Z}/\ell^{\mathrm{n}}(0)[-2]))$$

by (9.3). This is because the codimension of $\mathrm{X}\times\{0\}$ in $\mathrm{X}\times\mathbb{A}^1$ is 1. Now $\mathbb{Z}/\ell^n(0)$, which is the motivic complex of weight 0 identifies with the constant sheaf $\mathbb{Z}/\ell^n$ (given by the ring of integers modulo ℓ^n). The shift $[-2]$ shifts this sheaf to degree 2, so that $\mathbb{Z}/\ell^n(0)[-2]$ is the complex of Zariski sheaves concentrated in degree 2, where it is $\mathbb{Z}/\ell^n$. Since constant sheaves are flabby on the Zariski site of any irreducible scheme, the Zariski cohomology of $\mathrm{X}\times\{0\}$ with respect to this complex is trivial in degree 3 or higher. This observation completes the proof that $\mathrm{H}^{1,0}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}}) \cong 0$. and complete the proof of the remaining statements in (i).

The statements in (ii) and (iii) on etale cohomology may be proven using ascending induction on q. One may also want to observe that $\mathrm{H}^{\mathrm{i}}_{\mathrm{et}}(\mathrm{Y}, \mu_{\ell^{\mathrm{n}}}(\mathrm{j})) = 0$, for all $i < 0$.

The last statement now follows from (i) and (ii), as well as the identification of the Brauer group $\mathrm{Br}(\mathrm{Y})_{\ell^{\mathrm{n}}}$ for a smooth scheme Y with the cokernel of the cycle map $\mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{Y})_{\ell^{\mathrm{n}}} \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{Y}, \mu_{\ell^{\mathrm{n}}}(1))$ as observed in the discussion following (1.5). □

2.1. **External product pairing and the computation of the Brauer group of a split torus.** We will assume that X and Z are smooth schemes over B; and we will further assume $\mathcal{O}_{\mathrm{B}}$ has ℓ^n-th roots of unity. We first consider the following external product pairing, where $\mu_{\ell^{\mathrm{n}}}(0)$ is identified with the constant sheaf $\mathbb{Z}/\ell^n$:

$$\mathbf{x} : \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_{\ell^{\mathrm{n}}}(1)) \otimes \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(0)) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{Z}\times\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(1)) \tag{2.1}$$

One can also interpret the latter in terms of the following well-known construction of cyclic algebras. Assume that the structure sheaf of the scheme Z has the following property: $\mathrm{Pic}(\mathrm{Z}) \cong 0$, which implies the boundary map

$$\delta : cokernel(\Gamma(\mathrm{Z}, \mathbb{G}_{\mathrm{m}}) \stackrel{\ell^{\mathrm{n}}}{\to} \Gamma(\mathrm{Z}, \mathbb{G}_{\mathrm{m}})) \to \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_{\ell^{\mathrm{n}}}(1))$$

is an isomorphism. Let $a \in \Gamma(\mathrm{Z}, \mathbb{G}_{\mathrm{m}})$ and let $\mathrm{Y} \to \mathrm{X}$ denote a $\mathbb{Z}/\ell^n$-torsor corresponding to a class in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(0))$. Let σ denote the generator of $Aut_{\mathrm{X}}(\mathrm{Y})) \cong \mathbb{Z}/\ell^{\mathrm{n}}$. Associated to Y and the class a (identified with $a \otimes 1 \in \mathcal{O}_{\mathrm{Z}} \otimes \mathcal{O}_{\mathrm{Y}} \cong \mathcal{O}_{\mathrm{Z}\times\mathrm{Y}}$), one defines the *cyclic algebra* $\mathcal{O}_{\mathrm{Z}\times\mathrm{Y}}[x]_\sigma/(x^{\ell^n} - a)$, where $x.y' = \sigma(y').x$, for all $y' \in \mathcal{O}_{\mathrm{Y}\times\mathrm{Z}}$. This defines a class in $\mathrm{Br}(\mathrm{Z}\times\mathrm{X})_{\ell^{\mathrm{n}}}$ and identifies with the class defined as the image of Y and $\delta(a) \in \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_{\ell^{\mathrm{n}}}(1))$ under the external product pairing in (2.1).

Next we take $\mathrm{Z} = \mathbb{G}_{\mathrm{m}}$, the multiplicative group scheme defined over B, Now $\mathcal{O}_{\mathbb{G}_m} = \mathcal{O}_{\mathrm{B}}[t, t^{-1}]$. Let $\mathrm{Y} \to \mathrm{X}$ denote a $\mathbb{Z}/\ell^n$-torsor corresponding to a class in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(0))$ as in the last paragraph. Then, one may verify that the mapping $\mathrm{Y} \mapsto \mathcal{O}_{\mathrm{Y}\times\mathbb{G}_{\mathrm{m}}}[x]_\sigma/(x^{\ell^n} - t)$, is an injection $\mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{Br}(\mathrm{X}\times\mathbb{G}_{\mathrm{m}})_{\ell^{\mathrm{n}}}$, with inverse defined by the residue map associated to the divisor obtained by setting $t = 1$ in $\mathbb{G}_m$: see [CTS, p. 32]. (To be able to invoke [CTS, p. 32], one needs to first pull back classes in $\mathrm{H}^1_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}}, \mu_{\ell^{\mathrm{n}}}(1))$ and in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(0))$ to classes in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{K}(\mathbb{G}_{\mathrm{m}}\times\mathrm{X}), \mu_{\ell^{\mathrm{n}}}(1))$ and $\mathrm{H}^1_{\mathrm{et}}(\mathrm{K}(\mathbb{G}_{\mathrm{m}}\times\mathrm{X}), \mu_{\ell^{\mathrm{n}}}(0))$. Observe that the pull-back $p_2^* : \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_\ell(0)) \to \mathrm{H}^1_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}}\times\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(0)) \to \mathrm{H}^1_{\mathrm{et}}(\mathrm{K}(\mathbb{G}_{\mathrm{m}}\times\mathrm{X}), \mu_{\ell^{\mathrm{n}}}(0))$ is an injection.)

Next one may take $\mathrm{X} = \mathbb{G}_{\mathrm{m}}$ to obtain the external product pairing:

$$\mathbf{x} : \mathrm{H}^1_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}}, \mu_{\ell^{\mathrm{n}}}(1)) \otimes \mathrm{H}^1_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}}, \mu_{\ell^{\mathrm{n}}}(0)) \rightarrow \mathrm{H}^2_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}, \mu_{\ell^{\mathrm{n}}}(1)). \tag{2.2}$$

We proceed to interpret this pairing also in terms of cyclic algebras, under the assumption the base scheme B has the property that ℓ is invertible in $\mathcal{O}_{\mathrm{B}}$ and that it has a primitive ℓ^n-th root of unity ζ. Therefore, the sheaf μ_{ℓ^n} identifies the with the constant sheaf $\mathbb{Z}/\ell^n$. Given a unit $b \in \Gamma(\mathbb{G}_m, \mathbb{G}_m)$, let $\mathrm{Y} \rightarrow \mathbb{G}_{\mathrm{m}}$ denote the $\mathbb{Z}/\ell^n$-torsor given by $\mathrm{Spec}\,(\mathcal{O}_{\mathbb{G}_m}[x]/(x^{\ell^n} - b)) \rightarrow \mathbb{G}_m$: we equip this torsor with the automorphism σ given by sending $x \mapsto x\zeta$. Therefore, given two units $a, b \in \Gamma(\mathbb{G}_m, \mathbb{G}_m)$, one may define a cyclic algebra $(a, b)_\zeta$, by applying the construction in the last paragraph with $\mathrm{X} = \mathbb{G}_{\mathrm{m}}$, and the torsor $\mathrm{Y} \rightarrow \mathrm{X}$ given by the torsor $\mathrm{Spec}\,(\mathcal{O}_{\mathrm{X}}[x]/(x^{\ell^n} - b)) \rightarrow \mathrm{X} = \mathbb{G}_{\mathrm{m}}$.

At this point if X is any smooth scheme of finite type over B, pre-composing the external product pairing in (2.2) with the cup-product with $\mathrm{H}^0_{\mathrm{et}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}})$ defines classes in $\mathrm{H}^2_{\mathrm{et}}(\mathrm{X} \times \mathbb{G}^2_{\mathrm{m}}; \mu_{\ell^{\mathrm{n}}}(1))$, and hence classes in $\mathrm{Br}(\mathrm{X} \times \mathbb{G}^2_{\mathrm{m}})_{\ell^{\mathrm{n}}}$. In terms of cyclic algebras this corresponds to letting $a = t_1$ and $b = t_2$ in the discussion in the last paragraph, and where $\mathcal{O}_{\mathbb{G}^2_m} = \mathcal{O}_{\mathrm{B}}[t_1, t_2, t_1^{-1}, t_2^{-1}]$. This defines the summand $(\oplus^2 \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(0))) \oplus \mathrm{H}^0_{\mathrm{et}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}})$ in $\mathrm{Br}(\mathrm{X} \times \mathbb{G}^2_{\mathrm{m}})_{\ell^{\mathrm{n}}})$ appearing on the right-hand-side of Corollary 2.3(iii) with $q = 2$.

On considering $\mathbb{G}^q_m$, for $q \geq 2$, one may iterate the external product pairing in (2.2) and interpret that in terms of the cyclic algebras $(t_i, t_j)_\zeta$, $i < j$, where $\mathcal{O}_{\mathbb{G}^q_m} = \mathcal{O}_{\mathrm{B}}[t_1, \cdots, t_q, t_1^{-1}, \cdots, t_q^{-1}]$. One may then precompose these cup-products with the cup-product with $\mathrm{H}^0_{\mathrm{et}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}})$ to obtain classes in $\mathrm{Br}(\mathrm{X} \times \mathbb{G}^{\mathrm{q}}_{\mathrm{m}})_{\ell^{\mathrm{n}}}$. One may see that these cyclic algebras correspond to the $\binom{q}{2}$ summands of $\mathrm{H}^0_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}}}(0))$ appearing on the right-hand-side of Corollary 2.3(iii).

2.2. **Localization sequences and the residue map.** Next we consider localization sequences for motivic and étale cohomology theories, which provide a convenient technique for computing Brauer groups.

Proposition 2.4. *We start by considering the following situation. Consider X a smooth scheme of finite type over a base scheme* B *as in* (1.1) *which is of pure dimension over* B*. Fix a closed smooth subscheme Z in* X *that is of pure codimension in* X *with open complement U. Then one obtains a commutative diagram*

$$\begin{array}{ccccccccc}
\mathrm{H}^{1,1}_{\mathrm{M}}(\mathrm{U}, \mathbb{Z}/\ell^{\mathrm{n}}) & \rightarrow & \mathrm{H}^{2,1}_{\mathrm{M,Z}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}}) & \rightarrow & \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}}) & \rightarrow & \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{U}, \mathbb{Z}/\ell^{\mathrm{n}}) & \rightarrow & \\
\downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle a_3} & & \downarrow{\scriptstyle a_4} & & \\
\mathrm{H}^1_{\mathrm{et}}(\mathrm{U}, \mu_{\ell^{\mathrm{n}}}(1)) & \rightarrow & \mathrm{H}^2_{\mathrm{et,Z}}(\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(1)) & \rightarrow & \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(1)) & \rightarrow & \mathrm{H}^2_{\mathrm{et}}(\mathrm{U}, \mu_{\ell^{\mathrm{n}}}(1)) & \rightarrow &
\end{array}$$

$$\begin{array}{ccccc}
\rightarrow & \mathrm{H}^{3,1}_{\mathrm{M,Z}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}}) & \rightarrow & \mathrm{H}^{3,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}}) \\
 & \downarrow{\scriptstyle a_5} & & \downarrow{\scriptstyle a_6} \\
\rightarrow & \mathrm{H}^3_{\mathrm{et,Z}}(\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(1)) & \rightarrow & \mathrm{H}^3_{\mathrm{et}}(\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(1))
\end{array}$$

so that the following hold:

(1) The maps a_1 and a_2 are isomorphisms always (under our hypotheses as in (1.1)*.*

(2) The maps a_3 and a_4 are always injective.

(3) Under the assumption that Z *is also smooth and of pure codimension* 1*, we obtain isomorphisms* $\mathrm{H}^{2,1}_{\mathrm{M,Z}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}})) \cong \mathrm{H}^{0,0}_{\mathrm{M}}(\mathrm{Z}, \mathbb{Z}/\ell^{\mathrm{n}}))$ *and* $H^2_{et,\mathrm{Z}}(\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(1))) \cong \mathrm{H}^0_{\mathrm{et}}(\mathrm{Z}, \mu_{\ell^{\mathrm{n}}}(0))$.

(4) $\mathrm{H}^{3,1}_{\mathrm{M,Z}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}}) \cong 0$.
(5) $\mathrm{H}^{3,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}}) = 0$.
(6) If Z *has pure codimension* > 1*, then* $\mathrm{H}^{3}_{\mathrm{et,Z}}(\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(1)) \cong 0$ *as may be seen from cohomological purity.*

Proof. The fact that the maps a_3 and a_4 are always injective follows from the Kummer sequence considered in (1.5). Observe that the scheme X is assumed to be smooth. The isomorphisms in the third statement are essentially Thom-isomorphisms, which exist as the schemes X and Z are assumed to be smooth.

The isomorphism $\mathrm{H}^{3,1}_{\mathrm{M,Z}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}}) \cong 0$ in statement (4) may be obtained as follows. First observe that the motivic complex $\mathbb{Z}/\ell^n(1)$ is concentrated in degrees 0 and 1: in fact it is the complex $\mathbb{G}_m \overset{\ell^n}{\rightarrow} \mathbb{G}_m$ concentrated in degrees 0 and 1. The localization sequence in (9.2) shows that $\mathrm{H}^{3,1}_{\mathrm{M,Z}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}})$ identifies with the cohomology of the complex $i_*(\mathbb{Z}^{\mathrm{Z}}(1-c)[-2c]$ computed on the Zariski site of the base scheme X. Here c is the codimension of Z in X. If $c > 1$, clearly this complex is trivial and hence conclusion follows in this case. If $c = 1$, Thom isomorphism provides the isomorphism: $\mathrm{H}^{3,1}_{\mathrm{M,Z}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}}) \cong \mathrm{H}^{1,0}(\mathrm{Z}, \mathbb{Z}/\ell^{\mathrm{n}})$, which can be shown to be trivial as in Corollary 2.3.

The vanishing of the cohomology in the statement (5) may be obtained as follows. First, when the base scheme B is a field, this follows from the identification of the motivic cohomology $\mathrm{H}^{3,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}})$ with $\mathrm{CH}^{1,2-3}((\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}})$ which is trivial for obvious reasons. In general, as is shown in [Geis, Corollary 3.3], one takes the motivic complex $\mathbb{Z}/\ell^n(1)$, viewed as a complex of sheaves on the (small) flat site of the given scheme X and takes its pushforward to the Zariski site of the base scheme B, and then computes the Zariski cohomology of the resulting complex on B. The complex $\mathbb{Z}/\ell^n(1)$ is concentrated in degrees 0 and 1 and as B has Zariski cohomological dimension at most 1, the resulting complex on the Zariski site of B has no cohomology in degree 3 or above, proving the vanishing of the cohomology in the last statement. Moreover the vanishing of the local cohomology in the last statement is clear.

Therefore, it suffices to prove the first statement, which we proceed to do presently. We will first consider the map a_2. The localization sequence as in (9.2) readily provides the identification:

$$\mathrm{H}^{2,1}_{\mathrm{M,Z}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}}) \cong \mathrm{H}^{0,0}_{\mathrm{M}}(\mathrm{Z}, \mathbb{Z}/\ell^{\mathrm{n}}), \text{ while} \tag{2.3}$$

Thom isomorphism (or purity) provides the isomorphism:

$$\mathrm{H}^{2}_{\mathrm{et,Z}}(\mathrm{X}, \mu_{\ell^{\mathrm{n}}}) \cong \mathrm{H}^{0}_{\mathrm{et}}(\mathrm{Z}, \mu_{\ell^{\mathrm{n}}}(0)). \tag{2.4}$$

Clearly the right-hand-sides of both (2.3) and (2.4) are isomorphic to the sum $\oplus\mathbb{Z}/\ell^n$ indexed by the connected components of Z and are therefore isomorphic, thereby proving that the map a_2 is an isomorphism.

Then the Beilinson-Lichtenbaum conjecture (which follows as a consequence of the *motivic Bloch-Kato conjecture*) first shows that the map a_1 is an isomorphism, when the base scheme B is the spectrum of a field. Observe that, this is the statement that the natural map

$$\mathbb{Z}/\ell^n(i) \simeq \tau_{\leq i} R\epsilon_* \epsilon^*(\mathbb{Z}/\ell^n(i)) \simeq \tau_{\leq i} R\epsilon_*(\mu_{\ell^n}(i)), \tag{2.5}$$

where ϵ is the obvious map of sites from the big étale site of the scheme B to the corresponding big Nisnevich site. It is shown in [Spitz, Theorem 3.9] that the quasi-isomorphism in (2.5) extends to the case where B is a Dedekind domain. □

Corollary 2.5. *Assume* X *is a smooth scheme of pure dimension over* B*, with* Z *a closed smooth subscheme of pure codimension in* X *with open complement* U*. Then, if* Z *has pure codimension* 1 *in* X*, we obtain the exact sequence:*

$$0 \to \mathrm{Br(X)}_{\ell^n} \to \mathrm{Br(U)}_{\ell^n} \overset{res}{\longrightarrow} \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_{\ell^n}(0)).$$

In case the Z *has pure codimension* > 1 *in* X*, we obtain:*

$$\mathrm{Br(X)}_{\ell^n} \overset{\cong}{\to} \mathrm{Br(U)}_{\ell^n}.$$

Proof. First we consider the case when Z has pure codimension 1 in X. Then, one observes the Thom isomorphism (in view of the assumption that Z is also smooth):

$$\mathrm{H}^3_{\mathrm{et,Z}}(\mu_{\ell^n}(1) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_{\ell^n}(0)).$$

Next, one invokes Proposition 2.4 and Lemma 2.1 with $\alpha = a_1$, $\beta = a_2$, $\gamma = a_3$, $\delta = a_4$ and $\eta = a_5$ to obtain the exact sequence stated in the corollary. Observe that the resulting map

$$\mathrm{Br(U)}_{\ell^n} \longrightarrow \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_{\ell^n}(0))$$

identifies with the residue map discussed in section 2.4. (Hence we denote this map by *res*.)

In case the codimension of Z in X is larger than 1, the same Proposition 2.4 and Lemma 2.1 provides the second conclusion. □

2.3. **Computing Equivariant Brauer groups.** In this section, we will discuss certain techniques that will facilitate the computation of equivariant Brauer groups. We will make use of the admissible gadgets defined in Example 3.3.

Let $\mathrm{G} = \mathrm{GL_n}$, for a fixed integer $n > 0$. Let $\mathrm{W} = \mathrm{End}(\mathbb{A}^n) =$ the space of all $n \times n$-matrices with entries in B. In this case, we will let

$$\mathrm{EG}^{\mathrm{gm},2} = (\mathrm{GL_n} \times \mathrm{W}) \bigcup_{\mathrm{GL_n} \times \mathrm{GL_n}} (\mathrm{W} \times \mathrm{GL_n}). \tag{2.6}$$

The determination of the *bad set* $\mathrm{Z_m}$ as in Example 3.3 shows that the codimension of $\mathrm{EG}^{\mathrm{gm},2}$ in $\mathrm{W} \times \mathrm{W}$ is 2. We also observe that the following hold, when X is a scheme provided with an action by GL_n:

(i) $(\mathrm{GL}_n \times \mathrm{W}) \times_{\mathrm{GL_n}} \mathrm{X}$ is open in $\mathrm{EG}^{\mathrm{gm},2} \times_{\mathrm{GL_n}} \mathrm{X}$ with the compliment being $((\mathrm{W} - \mathrm{GL_n}) \times \mathrm{GL_n}) \times_{\mathrm{GL_n}} \mathrm{X}$

(ii) $((\mathrm{W} - \mathrm{GL_n}) \times \mathrm{GL_n}) \times_{\mathrm{GL_n}} \mathrm{X}$ has codimension 1 in $(\mathrm{GL}_n \times \mathrm{W}) \times_{\mathrm{GL_n}} \mathrm{X}$.

Therefore, Corollary 2.5 provides the short exact sequence:

$$0 \to \mathrm{Br}_{\mathrm{GL_n}}(\mathrm{X})_{\ell^n} \to \mathrm{Br}((\mathrm{GL_n} \times \mathrm{W}) \times_{\mathrm{GL_n}} \mathrm{X})_{\ell^n} \overset{res}{\longrightarrow} \mathrm{H}^1_{\mathrm{et}}((\mathrm{W} - \mathrm{GL_n}) \times \mathrm{GL_n}) \times_{\mathrm{GL_n}} \mathrm{X}, \mu_{\ell^n}(0)). \tag{2.7}$$

Clearly

$$(\mathrm{GL}_n \times \mathrm{W}) \times_{\mathrm{GL_n}} \mathrm{X} \cong \mathrm{W} \times \mathrm{X} \text{ and } ((\mathrm{W} - \mathrm{GL_n}) \times \mathrm{GL_n}) \times_{\mathrm{GL_n}} \mathrm{X} \cong (\mathrm{W} - \mathrm{GL_n}) \times \mathrm{X}, \tag{2.8}$$

so that the exact sequence (2.7) identifies with

$$0 \to \mathrm{Br}_{\mathrm{GL_n}}(\mathrm{X})_{\ell^n} \to \mathrm{Br(X)}_{\ell^n} \overset{\beta}{\to} \mathrm{H}^1_{\mathrm{et}}((\mathrm{W} - \mathrm{GL_n}) \times \mathrm{X}, \mu_{\ell^n}(0)). \tag{2.9}$$

However, *it may be important to point out that the map* β *can only be understood as the map corresponding to the residue map res under the above identifications.* We may also consider the following special cases of the above general result.

(i) Take $n = 1$, so that $\mathrm{GL}_n = \mathbb{G}_m$. In this case one may take $\mathrm{W} = \mathbb{A}^1$, so that $\mathrm{EG}^{\mathrm{gm},2} = \mathbb{A}^2 - \{0\} = \mathbb{A}^1 \times \mathbb{G}_\mathrm{m} \bigcup \mathbb{G}_\mathrm{m} \times \mathbb{A}^1$. In this case the exact sequence (2.9) becomes

(2.10)

$$0 \to \mathrm{Br}_{\mathbb{G}_\mathrm{m}}(\mathrm{X})_{\ell^\mathrm{n}} \to \mathrm{Br}((\mathbb{G}_\mathrm{m}\times\mathbb{A}^1)\times_{\mathbb{G}_\mathrm{m}}\mathrm{X})_{\ell^\mathrm{n}} \cong \mathrm{Br}(\mathrm{X})_{\ell^\mathrm{n}} \overset{res}{\to} \mathrm{H}^1_{\mathrm{et}}((\mathbb{A}^1-\mathbb{G}_\mathrm{m})\times\mathbb{G}_\mathrm{m})\times_{\mathbb{G}_\mathrm{m}}\mathrm{X}, \mu_{\ell^\mathrm{n}}(0)) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_{\ell^\mathrm{n}}(0)).$$

(ii) Take $n = 2$, so that in this case $\mathrm{W} = \mathrm{End}(\mathbb{A}^2)$ and $\mathrm{W} - \mathrm{GL}_2 = \overline{\mathrm{St}_1}(\mathbb{A}^2)$, where $\mathrm{St}_1(\mathbb{A}^2)$ denotes the Stiefel variety of 1-planes in $\mathbb{A}^2$, and $\overline{\mathrm{St}_1}(\mathbb{A}^2)$ denotes its closure in $\mathbb{A}^2$. Observe that $\mathrm{St}_1(\mathbb{A}^2) = \mathbb{A}^2 - \{0\}$, so that $\overline{\mathrm{St}_1}(\mathbb{A}^2) = \mathbb{A}^2$. In this case the exact sequence (2.9) becomes

$$0 \to \mathrm{Br}_{\mathrm{GL}_2}(\mathrm{X})_{\ell^\mathrm{n}} \to \mathrm{Br}(\mathrm{X})_{\ell^\mathrm{n}} \overset{\beta}{\to} \mathrm{H}^1_{\mathrm{et}}(\overline{\mathrm{St}_1}(\mathbb{A}^2) \times \mathrm{X}, \mu_{\ell^\mathrm{n}}(0)) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_{\ell^\mathrm{n}}(0)). \tag{2.11}$$

with the map β interpreted in terms of the map α in (2.7) making use of the identifications in (2.8).

(iii) In case G is a closed smooth affine group-scheme of GL_n, with G acting on X, one obtains an induced action by GL_n on $\mathrm{GL}_n \times_\mathrm{G} \mathrm{X}$. In view of Theorem 1.4, $\mathrm{Br}_\mathrm{G}(\mathrm{X})_{\ell^\mathrm{n}} \cong \mathrm{Br}_{\mathrm{GL_n}}(\mathrm{GL_n} \times_\mathrm{G} \mathrm{X})$. Therefore, in this case the exact sequence (2.9) becomes

$$0 \to \mathrm{Br}_\mathrm{G}(\mathrm{X})_{\ell^\mathrm{n}} \to \mathrm{Br}(\mathrm{GL_n} \times_\mathrm{G} \mathrm{X})_{\ell^\mathrm{n}} \to \mathrm{H}^1_{\mathrm{et}}((\mathrm{W} - \mathrm{GL_n}) \times (\mathrm{GL_n} \times_\mathrm{G} \mathrm{X}), \mu_{\ell^\mathrm{n}}(0)). \tag{2.12}$$

As an *example* of this, assume X is provided by an action of SL_n which in fact extends to an action by GL_n. Then $\mathrm{GL}_n \times_{\mathrm{SL}_n} \mathrm{X} \cong \mathrm{GL_n}/\mathrm{SL_n} \times \mathrm{X} \cong \mathbb{G}_\mathrm{m} \times \mathrm{X}$. Therefore, the exact sequence in (2.13) becomes

$$0 \to \mathrm{Br}_{\mathrm{SL_n}}(\mathrm{X})_{\ell^\mathrm{n}} \to \mathrm{Br}(\mathbb{G}_\mathrm{m} \times \mathrm{X})_{\ell^\mathrm{n}} \to \mathrm{H}^1_{\mathrm{et}}((\mathrm{W} - \mathrm{GL_n}) \times (\mathbb{G}_\mathrm{m} \times \mathrm{X}), \mu_{\ell^\mathrm{n}}(0)). \tag{2.13}$$

In particular, it follows that $\mathrm{Br}_{\mathrm{SL_n}}(\mathrm{X})_{\ell^\mathrm{n}}$ injects into $\mathrm{Br}(\mathbb{G}_\mathrm{m} \times \mathrm{X})_{\ell^\mathrm{n}} \cong \mathrm{Br}(\mathrm{X})_{\ell^\mathrm{n}} \oplus \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_{\ell^\mathrm{n}}(0))$.

(iv) We next consider the case where $\mathrm{G} = \mathbb{G}_\mathrm{m}^r$, a split torus of rank r, or more generally a diagonalizable group of the form $\mu_{n_1} \times \cdots \times \mu_{n_s} \times \mathbb{G}_m^t$, with $r = s+t$. In this case, we will always choose $\mathrm{EG}^{\mathrm{gm},2} = (\mathbb{A}^2 - \{0\})^r$. Moreover, we will also observe that when $s = r$, and $\mathbb{G}_m^r$ is provided with an action on the scheme X with an induced action by G on X, the quotient $(\mathbb{A}^2 - \{0\})^r \times_\mathrm{G} \mathrm{X}$ identifies with a sum of s line bundles over $(\mathbb{A}^2 - \{0\})^r \times_{\mathbb{G}_m^r} \mathrm{X}$ with their zero section removed. These observations will be very useful when we consider the Brauer groups of toric stacks.

Example 2.6. We will assume the base B is the spectrum of a field k of characteristic different from 2. We show here as an immediate consequence of (2.11) above that if $\bar{\mathcal{H}}_g$ denotes the moduli stack of stable hyper-elliptic curves of genus $g \geq 2$ and *even*, then $\mathrm{Br}(\bar{\mathcal{H}}_\mathrm{g})_{\ell^\mathrm{n}} \cong \mathrm{Br}(\mathrm{Spec}\, k)_{\ell^\mathrm{n}}$, for any ℓ different from 2 and invertible in the base field k. As observed in [LP, Lemma A.3], there is an open substack $\bar{\mathcal{H}}'_g \subseteq \bar{\mathcal{H}}_g$ so that $\bar{\mathcal{H}}'_g = [\mathrm{U_g}/\mathrm{GL}_2]$, for an open subscheme $\mathrm{U_g} \subseteq \mathbb{A}^{2\mathrm{g}+3}$ so that the complement of $\mathrm{U_g}$ in $\mathbb{A}^{2g+3}$ has codimension greater than 1. Then the restriction $\mathrm{Br}(\bar{\mathcal{H}}_\mathrm{g})_{\ell^\mathrm{n}} \to \mathrm{Br}(\bar{\mathcal{H}}'_\mathrm{g})_{\ell^\mathrm{n}}$ is injective. Now $\mathrm{Br}(\bar{\mathcal{H}}'_\mathrm{g})_{\ell^\mathrm{n}} = \mathrm{Br}([\mathrm{U_g}/\mathrm{GL}_2])_{\ell^\mathrm{n}}$. By (2.11), the latter injects into $\mathrm{Br}(\mathrm{U_g})_{\ell^\mathrm{n}} \cong \mathrm{Br}(\mathbb{A}^{2\mathrm{g}+3})_{\ell^\mathrm{n}} \cong \mathrm{Br}(\mathrm{Spec}\, k)_{\ell^\mathrm{n}}$. This shows the composite map above $\mathrm{Br}(\bar{\mathcal{H}}_\mathrm{g})_{\ell^\mathrm{n}} \to \mathrm{Br}(\mathbb{A}^{2\mathrm{g}+3})_{\ell^\mathrm{n}} \cong \mathrm{Br}(\mathrm{Spec}\, k)_{\ell^\mathrm{n}}$ is an injection.

On the other hand, one has the pull-back $\pi^* : \mathrm{Br}(\mathrm{Spec}\, k)_{\ell^\mathrm{n}} \to \mathrm{Br}(\bar{\mathcal{H}}_\mathrm{g})_{\ell^\mathrm{n}}$: the composition of this map with the above map $\mathrm{Br}(\bar{\mathcal{H}}_\mathrm{g})_{\ell^\mathrm{n}} \to \mathrm{Br}(\mathbb{A}^{2\mathrm{g}+3})_{\ell^\mathrm{n}} \cong \mathrm{Br}(\mathrm{Spec}\, k)_{\ell^\mathrm{n}}$ is clearly an isomorphism. This shows that the map $\mathrm{Br}(\bar{\mathcal{H}}_\mathrm{g})_{\ell^\mathrm{n}} \to \mathrm{Br}(\mathbb{A}^{2\mathrm{g}+3})_{\ell^\mathrm{n}} \cong \mathrm{Br}(\mathrm{Spec}\, k)_{\ell^\mathrm{n}}$ is in fact an isomorphism.

2.4. **More on the residue map.** Assume the situation of Corollary 2.5. Then the resulting map

$$\mathrm{Br}(\mathrm{U})_{\ell^\mathrm{n}} \to \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_{\ell^\mathrm{n}}(0)) \tag{2.14}$$

has an explicit description in terms of *the residue map*, discussed in detail in [GS, Chapter 6] and also [CTS, Chapter I, 1.4], which we will summarize here.

First one observes that if p denotes the ideal sheaf in $\mathcal{O}_{\rm X}$ defining the divisor Z, then the localization of $\mathcal{O}_{\rm X}$ at p is a sheaf of discrete valuation rings. If K(X) denotes the corresponding function field of X, and k(Z) denotes the function field of Z, then one observes the following:

(i) the field $k({\rm Z}) \cong \mathcal{O}_{\rm X,p}/{\rm p}\mathcal{O}_{\rm X,p}$ and the latter identifies with the residue field of the corresponding valuation on $\mathcal{O}_{\rm X}$.

Throughout the remaining statements let ℓ denote a fixed prime different from the residue characteristics.

(ii) As X is smooth, the natural map ${\rm Br(X)}_{\ell^{\rm n}} \to {\rm Br(K(X))}_{\ell^{\rm n}}$ is an injection: see [CTS, Theorem 3.5.5].

(iii) As Z- (the generic point of Z) is closed and of dimension < the dimension of Z, the natural map ${\rm H}^1_{\rm et}({\rm Z}, \mu_{\ell^{\rm n}}) \to {\rm H}^1_{\rm et}({\rm Spec}\, k({\rm Z}), \mu_{\ell^{\rm n}})$ is also an injection.

(iv) A key result due to Merkurjev-Suslin identifies ${\rm K}_2^{Milnor}({\rm K(X)})/\ell^{\rm n}{\rm K}_2^{\rm Milnor}({\rm K(X)})$ with ${\rm H}^2_{\rm et}({\rm K(X)}, \mu_{\ell^{\rm n}}(2))$ and ${\rm K}_1^{Milnor}(k({\rm Z}))/\ell^{\rm n}{\rm K}_1^{\rm Milnor}(k({\rm Z}))$ with ${\rm H}^1_{\rm et}({\rm k(Z)}, \mu_{\ell^{\rm n}}(1))$. (See [MS], [GS, Chapter 8] and [CTS, Chapter 1, 1.4].)

(v) Another key result due to Merkurjev-Suslin shows that if the field K contains a primitive ℓ^n-th root of unity ζ, then any class in the Brauer group ${\rm Br(K)}_{\ell^{\rm n}}$ is *Brauer equivalent* to finite product of cyclic algebras of the form $(a_1, b_1)_\zeta \otimes_{\rm K} \cdots \otimes_{\rm K} (a_m, b_m)$, $a_i, b_i \epsilon {\rm K}^*$, $i = 1, \cdots, m$. (See [GS, Theorem 2.5.7].)

(vi) One also has the following commutative square (see [GS, Proposition 7.5.1]), which proves the compatibility of the *residue map* in étale cohomology with the *tame symbol map* on Milnor K-theory as in:

$$\begin{array}{ccc} {\rm K}_2^{Milnor}({\rm K(X)})/\ell^{\rm n}{\rm K}_2^{\rm Milnor}({\rm K(X)}) & \xrightarrow{Tame.sym} & {\rm K}_1^{Milnor}(k({\rm Z}))/\ell^{\rm n}{\rm K}_1^{\rm Milnor}({\rm k(Z)}) \\ \downarrow h^2 & & \downarrow h^1 \\ {\rm H}^2_{\rm et}({\rm K(X)}, \mu_{\ell^{\rm n}}(2)) & \xrightarrow{res} & {\rm H}^1_{\rm et}(k({\rm Z}), \mu_{\ell^{\rm n}}(1)) \end{array} \tag{2.15}$$

where the map denoted $Tame.sym$ is the *tame symbol* defined as follows:

$$\delta_v : {\rm K(X)}^* \times {\rm K(X)}^* \to k({\rm Z})^*, \delta_v({\rm a, b}) = (-1)^{v({\rm a}).v({\rm b})} \frac{{\rm a}^{v({\rm b})}}{{\rm b}^{v({\rm a})}} {\rm mod(p)} \tag{2.16}$$

where p denotes the ideal defining Z in X. Moreover, h^2 and h^1 are the *Galois symbol maps*: see [GS, Definition 4.6.4].

(vii) Finally we also make the observation that, assuming K(X) contains a primitive ℓ^n-th root of unity ζ, multiplication by ζ defines an isomorphism $\mu_{\ell^n}(i-1) \to \mu_{\ell^n}(i)$. Therefore, under this assumption and with the choice of a primitive ℓ^n-root of unity of ζ, the residue map above may be also viewed as a map

$${\rm H}^2_{\rm et}({\rm K(X)}, \mu_{\ell^{\rm n}}(1)) \xrightarrow{res} {\rm H}^1_{\rm et}(k({\rm Z}), \mu_{\ell^{\rm n}}(0)) \tag{2.17}$$

Moreover, in this case, the residue map has the following concrete interpretation:

$$residue((a, b)_\zeta) = \text{ the Galois extension of } k({\rm Z}) \text{ given by adjoining } (\frac{{\rm a}^{v_{\rm Z}({\rm b})}}{{\rm b}^{v_{\rm Z}({\rm a})}})^{1/\ell^{\rm n}}. \tag{2.18}$$

Remark 2.7. *It follows that in order to be able to apply the results above in (iv) through (vii), when one is considering the framework as in 1.1, one needs to consider schemes defined over* B[ζ]*, where* ζ *is a primitive* ℓ^n*-th root of unity and* ℓ *is different from the residue characteristics of* B.

3. Equivariant Brauer groups vs. Brauer groups of quotient stacks: Proof of Theorem 1.4

The goal of this section is to prove Theorem 1.4. We begin discussing the construction of geometric classifying spaces and Borel construction followed by the simplicial variant. Throughout this section, B will denote a Dedekind domain.

3.1. **Admissible gadgets.** Let G denote a fixed smooth affine group scheme over B. We will define a pair (W, U) of smooth schemes over B to be a *good pair* for G if W is a representation of G and $U \subsetneq W$ is a G-invariant non-empty open subscheme on which G acts freely and so that U/G is a quasi-projective scheme over B. Moreover, one may choose (W, U) so that the complement $W - U$ has arbitrarily high codimension. It is known (*cf.* [EG, Lemma 7, §6.2]) that a good pair for G always exists.

Definition 3.1. *A sequence of pairs* $\{(W_m, U_m)|m \geq 1\}$ *of smooth schemes over* B *is called an admissible gadget for* G*, if there exists a good pair* (W, U) *for* G *such that* $W_m = W^{\times^m}$ *and* $U_m \subsetneq W_m$ *is a* G*-invariant open subset such that the following hold for each* $m \geq 1$*.*

(1) $(U_m \times W) \cup (W \times U_m) \subseteq U_{m+1}$ *as* G*-invariant open subvarieties.*
(2) $\{\mathrm{codim}_{U_{m+1}}(U_{m+1} \setminus (U_m \times W))\,|m\}$ *is a strictly increasing sequence, that is,*
$$\mathrm{codim}_{U_{m+2}}(U_{m+2} \setminus (U_{m+1} \times W)) > \mathrm{codim}_{U_{m+1}}(U_{m+1} \setminus (U_m \times W)).$$
(3) $\{\mathrm{codim}_{W_m}(W_m \setminus U_m)\,|m\}$ *is a strictly increasing sequence, that is,*
$$\mathrm{codim}_{W_{m+1}}(W_{m+1} \setminus U_{m+1}) > \mathrm{codim}_{W_m}(W_m \setminus U_m).$$
(4) U_m *has a free* G*-action, the quotient* U_m/G *is a smooth quasi-projective scheme over* B *and* $U_m \to U_m/G$ *is a principal* G*-bundle.*
(5) In addition, we will also assume the following (see [MV, Definition 2.1, p. 133]*):*
$U_m(G)$ *has a k-rational point*

Lemma 3.2. *Let* U *denote a smooth quasi-projective scheme over* B *with a free action by the smooth affine group scheme* G *so that the quotient* U/G *exists as a smooth quasi-projective scheme over* B*. Then if* X *is any smooth* G*-quasi-projective scheme over* B*, the quotient* $U \underset{G}{\times} X \cong (U \underset{\mathrm{Spec}\,B}{\times} X)/G$ *(for the diagonal action of* G*) exists as a scheme over* B*.*

Proof. This follows, for example, from [MFK94, Proposition 7.1]. □

Example 3.3. *An example of an admissible gadget for* G *can be constructed as follows: start with a good pair* (W, U) *for* G*. The choice of such a good pair will vary depending on* G*. Choose a faithful representation* R *of* G *of dimension n, that is,* G *admits a closed immersion into* $GL(R)$*. Then* G *acts freely on an open subset* U *of* $W = R^{\oplus n} = \mathrm{End}(R)$ *so that* U/G *is a variety. (For e.g.* $U = GL(R)$*.) Let* $Z = W \setminus U$*.*

Given a good pair (W, U)*, we now let*

$$W_m = W^{\times m}, U_1 = U \text{ and } U_{m+1} = (U_m \times W) \cup (W \times U_m) \text{ for } m \geq 1. \tag{3.1}$$

Setting $\mathrm{Z_1 = Z}$ *and* $\mathrm{Z_{m+1} = W_{m+1} \setminus U_{m+1}}$ *for* $m \geq 1$, *one checks that* $\mathrm{W_m \setminus U_m = Z^m}$ *and* $\mathrm{Z_{m+1} = Z^m \times Z}$. *In particular,* $\mathrm{codim_{W_m}(W_m \setminus U_m) = m(codim_W(Z))}$. *Moreover,* $\mathrm{U}_m \to \mathrm{U}_m/\mathrm{G}$ *is a principal* G*-bundle and the quotient* $\mathrm{V_m = U_m/G}$ *exists as a smooth quasi-projective scheme.*

3.2. **The geometric and simplicial Borel constructions.** Given an admissible gadget $\{(\mathrm{W_m, U_m})|\mathrm{m} \geq 0\}$ for the linear algebraic group G and a G-scheme X, we define

$$\mathrm{EG^{gm,m} = U_m, \quad EG^{gm,m} \times_G X = U_m \times_G X, \quad BG^{gm,m} = U_m \times_G (Spec\,}k), \text{ and} \qquad (3.2)$$
$$\pi_m : \mathrm{EG^{gm,m} \times_G X \to BG^{gm,m}}.$$

The ind-scheme $\{\mathrm{EG^{gm,m} \times_G X|m \geq 0}\}$ is called the *geometric Borel construction*. We will often denote $\lim_{m\to\infty}\{\mathrm{EG^{gm,m} \times_G X|m \geq 0}\}$ by $\mathrm{EG^{gm} \times_G X}$. We next consider $\mathrm{EG \times_G X}$ which is the simplicial scheme defined by $\mathrm{G^n \times X}$ in degree n, and with the structure maps defined as follows:

$$\begin{aligned} d_i(g_0, \cdots, g_n, x) &= (g_1, \cdots, g_n, x), i = 0 \\ &= (g_1, \cdots, g_{i-1}.g_i, \cdots, g_n, x), 0 < i < n \\ &= (g_1, \cdots, g_{n-1}, g_n.x), i = n, \text{ and} \\ s_i(g_0, \cdots, g_{n-1}, x) &= (g_0, \cdots, g_{i-1}, e, g_i, \cdots, x) \end{aligned} \qquad (3.3)$$

where $g_i \in \mathrm{G}$, $x \in \mathrm{X}$, $g_{i-1}.g_i$ denotes the product of g_{i-1} and g_i in G, while $g_n.x$ denotes the product of g_n and x. e denotes the unit element in G. This is the *simplicial Borel construction.* Then we obtain the following identification, which is well-known.

Lemma 3.4. *One obtains an isomorphism:* $\mathrm{EG \times_G X \cong cosk_0^{[X/G]}(X)}$, *where* $cosk_0^{[X/G]}(\mathrm{X})$ *is the simplicial scheme defined in degree* n *by the* $(n+1)$*-fold fibered product of* X *with itself over the stack* [X/G], *with the structure maps of the simplicial scheme* $cosk_0^{[X/G]}(\mathrm{X})$ *induced by the above fibered products.*

For each fixed $m \geq 0$, we obtain the diagram of simplicial schemes (where p_1 is induced by the projection $\mathrm{EG^{gm,m} \times X \to X}$ and p_2 is induced by the projection $\mathrm{EG \times (EG^{gm,m} \times X) \to EG^{gm,m} \times X}$):

$$\begin{array}{ccc} & \mathrm{EG \times_G (EG^{gm,m} \times X)} & \\ {}^{\mathrm{p_1}}\swarrow & & \searrow^{\mathrm{p_2}} \\ \mathrm{EG \times_G X} & & \mathrm{EG^{gm,m} \underset{G}{\times} X} \end{array} \qquad (3.4)$$

G acts diagonally on $\mathrm{EG \times_G (EG^{gm,m} \times X)}$.

Proposition 3.5. *(i) The map*

$$\mathrm{p_1^* : H^1_{et}(EG \times_G X, \mathbb{G}_m) \to H^1_{et}(EG \times_G (EG^{gm,m} \times X), \mathbb{G}_m)} \textit{ and the map} \qquad (3.5)$$
$$\mathrm{p_2^* : H^1_{et}(EG^{gm,m} \times_G X, \mathbb{G}_m) \to H^1_{et}(EG \times_G (EG^{gm,m} \times X), \mathbb{G}_m)}, \textit{ for } \mathrm{m} > 1,$$

are isomorphisms.

(ii) The corresponding maps, for $m > 1$ *with* $\ell \neq char(k)$,

$$\mathrm{p_1^* : H^2_{et}(EG \times_G X, \mu_{\ell^n}(1)) \to H^2_{et}(EG \times_G (EG^{gm,m} \times X), \mu_{\ell^n}(1))}, \textit{ and} \qquad (3.6)$$
$$\mathrm{p_2^* : H^2_{et}(EG^{gm,m} \times_G X, \mu_{\ell^n}(1)) \to H^2_{et}(EG \times_G (EG^{gm,m} \times X), \mu_{\ell^n}(1))}$$

are isomorphisms.

Proof. The isomorphisms in (i) are rather involved, and therefore, we discuss the proof of (i) first. A key to the proof is the observation that, over the base scheme B which is a Dedekind domain, $\mathrm{H}^1_{\mathrm{et}}(\mathbb{A}^{\mathrm{n}}, \mathbb{G}_{\mathrm{m}}) \cong \mathrm{Pic}(\mathbb{A}^{\mathrm{n}}, \mathbb{G}_{\mathrm{m}}) \cong 0$, for any $n \geq 0$. We consider the Leray spectral sequences associated to the maps p_1 and p_2:

$$\begin{aligned} \mathrm{E}_2^{\mathrm{s,t}}(1) &= \mathrm{H}^{\mathrm{s}}_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \times \mathrm{X}, \mathrm{R}^{\mathrm{t}}\mathrm{p}_{1*}(\mathbb{G}_{\mathrm{m}})) \Longrightarrow \mathrm{H}^{\mathrm{s+t}}_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} (\mathrm{EG}^{\mathrm{gm,m}} \times \mathrm{X}), \mathbb{G}_{\mathrm{m}}) \text{ and} \\ \mathrm{E}_2^{\mathrm{s,t}}(2) &= \mathrm{H}^{\mathrm{s}}_{\mathrm{et}}(\mathrm{EG}^{\mathrm{gm,m}} \times_{\mathrm{G}} \times \mathrm{X}, \mathrm{R}^{\mathrm{t}}\mathrm{p}_{2*}(\mathbb{G}_{\mathrm{m}})) \Longrightarrow \mathrm{H}^{\mathrm{s+t}}_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} (\mathrm{EG}^{\mathrm{gm,m}} \times \mathrm{X}), \mathbb{G}_{\mathrm{m}}). \end{aligned} \tag{3.7}$$

Since $s, t \geq 0$, both spectral sequences converge strongly.

The stalks of $\mathrm{R}^{\mathrm{t}}\mathrm{p}_{2*}(\mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^{\mathrm{t}}(\mathrm{EG} \underset{\mathrm{Spec\,B}}{\times} (\mathrm{Spec\,A}), \mathbb{G}_{\mathrm{m}})$, where A denotes a strict Hensel ring. (Strictly speaking, in order to obtain the above identification, we need to make use of the simplicial topology as in [J02] or [J20, 5.4]. But we will ignore this rather subtle point for the rest of the discussion.) Since $\mathrm{EG} \cong \mathrm{cosk}_0^{\mathrm{Spec\,B}}(\mathrm{G})$, $\mathrm{EG} \underset{\mathrm{Spec\,B}}{\times} (\mathrm{Spec\,A}) \cong \mathrm{cosk}_0^{\mathrm{Spec\,A}}(\mathrm{G} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A})$ is a *smooth hypercover* of Spec A. Therefore, we obtain the isomorphism:

$$\mathrm{H}^{\mathrm{t}}_{\mathrm{et}}(\mathrm{EG} \underset{\mathrm{Spec\,B}}{\times} (\mathrm{Spec\,A}), \mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^{\mathrm{t}}_{\mathrm{smt}}(\mathrm{EG} \underset{\mathrm{Spec\,B}}{\times} (\mathrm{Spec\,A}), \mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^{\mathrm{t}}(\mathrm{Spec\,A}, \mathbb{G}_{\mathrm{m}}). \tag{3.8}$$

These groups are trivial for $t = 1$ (see, for example, [Mi, Lemma 4.10]). Therefore, it follows that

$$\mathrm{R}^{t}\mathrm{p}_{2*}(\mathbb{G}_{\mathrm{m}})_{\mathrm{Spec\,A}} \cong 0, \text{ for } \mathrm{t} = 1. \tag{3.9}$$

Next we observe the isomorphism, by taking $t = 0$ in (3.8):

$$\mathrm{p}_{2*}(\mathbb{G}_{\mathrm{m}})_{\mathrm{Spec\,A}} \cong \mathrm{H}^0(\mathrm{EG} \underset{\mathrm{Spec\,B}}{\times} (\mathrm{Spec\,A}), \mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^0(\mathrm{Spec\,A}, \mathbb{G}_{\mathrm{m}}), \tag{3.10}$$

where $\mathrm{p}_{2*}(\mathbb{G}_{\mathrm{m}})_{\mathrm{Spec\,A}}$ denotes the stalk of the sheaf $\mathrm{p}_{2*}(\mathbb{G}_{\mathrm{m}})$ at Spec A. Observing that $\mathbb{G}_m$ is in fact a sheaf on the flat site, and therefore also on the smooth site, it follows that there is a natural map of sheaves $\mathbb{G}_m \to \mathrm{p}_{2*}(\mathbb{G}_{\mathrm{m}})$, where the $\mathbb{G}_m$ on the left (on the right) denotes the sheaf $\mathbb{G}_m$ restricted to the étale site of $\mathrm{EG}^{\mathrm{gm,m}} \times_{\mathrm{G}} \mathrm{X}$ (the étale site of $\mathrm{EG} \times_{\mathrm{G}} (\mathrm{EG}^{\mathrm{gm,m}} \times \mathrm{X})$, respectively). The isomorphism in (3.10) shows this map induces an isomorphism stalk-wise. It follows that the natural map $\mathbb{G}_m \to \mathrm{p}_{2*}(\mathbb{G}_{\mathrm{m}})$ of sheaves on the étale site is an isomorphism. This provides the isomorphism:

$$\mathrm{E}_2^{1,0}(2) = \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG}^{\mathrm{gm,m}} \times_{\mathrm{G}} \mathrm{X}, \mathrm{p}_{2*}(\mathbb{G}_{\mathrm{m}})) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG}^{\mathrm{gm,m}} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}), \mathrm{m} > 0. \tag{3.11}$$

The stalks of $\mathrm{R}^{\mathrm{t}}\mathrm{p}_{1*}(\mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^{\mathrm{t}}(\mathrm{EG}^{\mathrm{gm,m}} \underset{\mathrm{Spec\,B}}{\times} (\mathrm{Spec\,A}), \mathbb{G}_{\mathrm{m}})$, where A denotes a strict Hensel ring, for all $t \geq 0$. Observe that this strict Hensel ring A is the stalk of the structure sheaf of $(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X})_{\mathrm{n}} = \mathrm{G}^{\mathrm{n}} \times \mathrm{X}$, at a geometric point. Hence it is a filtered direct limit $\lim_i \mathrm{A}_{\mathrm{i}}$, with each A_{i} regular.

To determine the groups $\mathrm{H}^{\mathrm{t}}(\mathrm{EG}^{\mathrm{gm,m}} \underset{\mathrm{Spec\,B}}{\times} (\mathrm{Spec\,A}), \mathbb{G}_{\mathrm{m}})$, we consider the long exact sequence (with $\mathrm{EG}^{\mathrm{gm,m}} = \mathrm{U}_{\mathrm{m}}$, which is assumed to be an open subscheme of the affine space $\mathbb{A}^m$, with $\mathrm{Z}_{\mathrm{m}} = \mathbb{A}^{\mathrm{m}} - \mathrm{U}_{\mathrm{m}}$):

$$\begin{aligned} \cdots \to \mathrm{H}^0_{\mathrm{et}, \mathrm{Z}_{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}}(\mathbb{A}^{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}, \mathbb{G}_{\mathrm{m}}) \to \mathrm{H}^0_{\mathrm{et}}(\mathbb{A}^{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}, \mathbb{G}_{\mathrm{m}}) \to \\ \xrightarrow{\alpha} \mathrm{H}^0_{\mathrm{et}}(\mathrm{U}_{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}, \mathbb{G}_{\mathrm{m}}) \to \mathrm{H}^1_{\mathrm{et}, \mathrm{Z}_{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}}(\mathbb{A}^{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}, \mathbb{G}_{\mathrm{m}}) \\ \to \mathrm{H}^1_{\mathrm{et}}(\mathbb{A}^{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{\beta} \mathrm{H}^1_{\mathrm{et}}(\mathrm{U}_{\mathrm{m}} \times_{\mathrm{Spec}\,k} \mathrm{Spec\,A}, \mathbb{G}_{\mathrm{m}}) \to \\ \to \mathrm{H}^2_{\mathrm{et}, \mathrm{Z}_{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}}(\mathbb{A}^{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}, \mathbb{G}_{\mathrm{m}}) \to \cdots \end{aligned} \tag{3.12}$$

Next we observe the identification of $\mathbb{G}_m$ with $\mathbb{Z}(1)[1]$ from Proposition 9.2 in the Appendix. As a result, we obtain the following identifications, for a smooth scheme Y of finite type over the base B, which is assumed to be a Dedekind domain (which also includes the case of it being a field) and a closed smooth subscheme Z of pure codimension c in Y:

$$\begin{aligned}\mathrm{H}^1_{\mathrm{et,Z}}(\mathrm{Y},\mathbb{G}_{\mathrm{m}}) &\cong \mathrm{H}^1_{\mathrm{Zar,Z}}(\mathrm{Y},\mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^{2,1}_{\mathrm{M,Z}}(\mathrm{Y}), \text{ and}\\ \mathrm{H}^0_{\mathrm{et,Z}}(\mathrm{Y},\mathbb{G}_{\mathrm{m}}) &\cong \mathrm{H}^0_{\mathrm{Zar,Z}}(\mathrm{Y},\mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^{1,1}_{\mathrm{M,Z}}(\mathrm{Y}).\end{aligned} \tag{3.13}$$

Therefore, by Proposition 9.4 in the Appendix, we see that if $c > 1$, then

$$\begin{aligned}\mathrm{H}^1_{\mathrm{et,Z}}(\mathrm{Y},\mathbb{G}_{\mathrm{m}}) &\cong \mathrm{H}^{2,1}_{\mathrm{M,Z}}(\mathrm{Y}) \cong 0, \text{ and}\\ \mathrm{H}^0_{\mathrm{et,Z}}(\mathrm{Y},\mathbb{G}_{\mathrm{m}}) &\cong \mathrm{H}^{1,1}_{\mathrm{M,Z}}(\mathrm{Y}) \cong 0.\end{aligned} \tag{3.14}$$

The map denoted α (β) in the long exact sequence (3.12) identifies with the restriction

$$\mathrm{H}^{1,1}_{\mathrm{M}}(\mathbb{A}^{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}) \to \mathrm{H}^{1,1}_{\mathrm{M}}(\mathrm{U}_{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A})$$

$$(\mathrm{H}^{2,1}_{\mathrm{M}}(\mathbb{A}^{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}) \to \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{U}_{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}), \text{respectively})$$

forming part of the localization sequence for the motivic cohomology groups. In fact, the corresponding localization sequence is given by:

$$\begin{aligned}&\cdots \to \mathrm{H}^{1,1-c_{\mathrm{m}}}_{\mathrm{M}}(\mathrm{Z}_{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}) \to \mathrm{H}^{1,1}_{\mathrm{M}}(\mathbb{A}^{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}) \xrightarrow{\alpha'} \mathrm{H}^{1,1}_{\mathrm{M}}(\mathrm{U}_{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A})\\ &\to \mathrm{H}^{2,1-c_{\mathrm{m}}}_{\mathrm{M}}(\mathrm{Z}_{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}) \to \mathrm{H}^{2,1}_{\mathrm{M}}(\mathbb{A}^{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}) \xrightarrow{\beta'} \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{U}_{\mathrm{m}} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}) \to 0\end{aligned} \tag{3.15}$$

where c_m denotes the codimension of $\mathrm{Z_m}$ in $\mathbb{A}^m$, which we assume satisfies $c_{\mathrm{m}} > 1$. To see that one gets such a localization sequence, one first replaces the strict Hensel ring A by one of the $\mathrm{A_i}$, where $\mathrm{A} = \lim_{\mathrm{i}} \mathrm{A_i}$, with each $\mathrm{A_i}$ a regular local ring. Clearly then the corresponding localization sequence exists and the groups in (3.15) involving the $\mathrm{Z_m}$ are trivial, as $c_{\mathrm{m}} > 1$, by assumption. At this point, one takes the direct limit over the $\mathrm{A_i}$: since the motivic cohomology groups are contravariantly functorial for flat maps, and filtered colimits are exact, we obtain the localization sequence (3.15). Moreover, the groups appearing in (3.15) involving the Z_m are all trivial, thereby proving that the maps α' and β' in (3.15), and therefore, the maps α and β in (3.12) are isomorphisms. This provides the isomorphisms for $t = 0, 1$:

$$\begin{aligned}\mathrm{R}^t\mathrm{p}_{1*}(\mathbb{G}_{\mathrm{m}})_{\mathrm{Spec\,A}} \cong \mathrm{H}^{\mathrm{t}}_{\mathrm{et}}(\mathrm{U}_{\mathrm{m}} \times_{\mathrm{Spec}\,k} \mathrm{Spec\,A}, \mathbb{G}_{\mathrm{m}}) &\cong \mathrm{H}^{\mathrm{t}}_{\mathrm{et}}(\mathbb{A}^{\mathrm{m}} \times_{\mathrm{Spec}\,k} \mathrm{Spec\,A}, \mathbb{G}_{\mathrm{m}})\\ &\cong \mathrm{H}^{\mathrm{t}}_{\mathrm{et}}(\mathrm{Spec\,A}, \mathbb{G}_{\mathrm{m}}).\end{aligned} \tag{3.16}$$

Therefore, it follows that

$$\mathrm{R}^t\mathrm{p}_{1*}(\mathbb{G}_{\mathrm{m}})_{\mathrm{Spec\,A}} \cong 0, \text{ for t} = 1. \tag{3.17}$$

Since $\mathbb{G}_m$ is a sheaf on the flat and hence on the smooth topology, there is a natural map $\mathbb{G}_m \to \mathrm{p}_{1*}(\mathbb{G}_{\mathrm{m}})$ of sheaves where the $\mathbb{G}_m$ on the left (on the right) is a sheaf on the étale site of $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}$ (on the étale site of $\mathrm{EG} \times_{\mathrm{G}} (\mathrm{EG}^{\mathrm{gm,m}} \times \mathrm{X})$, respectively). The stalk-wise isomorphism in (3.16) for $t = 0$ shows that the natural map $\mathbb{G}_m \to \mathrm{p}_{1*}(\mathbb{G}_{\mathrm{m}})$ of sheaves on the étale site is an isomorphism. This provides the isomorphism:

$$\mathrm{E}_2^{1,0}(1) = \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathrm{p}_{1*}(\mathbb{G}_{\mathrm{m}})) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}). \tag{3.18}$$

Moreover, observing that the differentials in the spectral sequences above go from $\mathrm{E}_{\mathrm{r}}^{\mathrm{p,q}}$ to $\mathrm{E}_{\mathrm{r}}^{\mathrm{p+r,q-r+1}}$, one sees (using (3.9) and (3.17)) that

$$\mathrm{E}_{\mathrm{r}}^{0,1}(1) = \mathrm{E}_{\mathrm{r}}^{0,1}(2) = 0 \text{ for all r} \geq 2 \text{ and that } \mathrm{E}_2^{1,0}(\mathrm{i}) \cong \mathrm{E}_{\mathrm{r}}^{1,0}(\mathrm{i}), \text{ for all r} \geq 2, \mathrm{i} = 1, 2. \tag{3.19}$$

The last observation shows that $\mathrm{E}_2^{1,0}(\mathrm{i})$, $i = 1, 2$ is isomorphic to the abutment in degree 1, namely, $\mathrm{H}^1_{\mathrm{et}}(\mathrm{EG}\times_{\mathrm{G}} (\mathrm{EG}^{\mathrm{gm,m}} \times \mathrm{X}), \mathbb{G}_{\mathrm{m}})$, $m > 1$. (Observe that the assumption $m > 1$ implies the codimension c_m of $\mathrm{Z_m}$ in $\mathbb{A}^m$ is at least 2.) Therefore, the isomorphisms in (3.18) and (3.11) complete the proof of (i).

Next we consider the proof of (ii). The key point is to consider the Leray spectral sequences for the maps p_1 and p_2. In this case, one may readily compute the stalks of $\mathrm{R^t p_{i*}}(\mu_{\ell^{\mathrm{n}}}(1))$ to be trivial for $t = 1, 2$ and $\cong \mu_{\ell^n}(1)$ for $t = 0$, and for $m >> 0$. (See [J20, Theorem 1.6] for further details.) Therefore, the conclusions in (ii) follow readily. □

Let S denote an algebraic stack, which we will assume is of *Artin type* and of finite type over the given base field k, with $x : \mathrm{X} \to S$ an *atlas*, that is, a *smooth surjective map from an algebraic space* X. We let $\mathrm{B}_x S = \mathrm{cosk}_0^S(\mathrm{X})$ denote the corresponding simplicial algebraic space. Then we let S_{smt} denote the smooth site, whose objects are $y : \mathrm{Y} \to S$, with y a smooth map from an algebraic space Y to S, and where a morphism between two such objects $y' : \mathrm{Y}' \to S$ and $y : \mathrm{Y} \to S$ is given by a map $f : \mathrm{Y}' \to \mathrm{Y}$ making the triangle

$$\begin{array}{ccc} \mathrm{Y}' & \xrightarrow{f} & \mathrm{Y} \\ & {\scriptstyle y'}\searrow \quad \swarrow{\scriptstyle y} & \\ & S & \end{array}$$

commute. The same definition defines the smooth site of any algebraic space. The smooth and étale sites of the simplicial algebraic space $\mathrm{B}_x S$ may be defined as follows. The objects of $Smt(\mathrm{B}_x S)$ are given by smooth maps $u_n : \mathrm{U_n} \to (\mathrm{B}_x S)_{\mathrm{n}}$ for some $n \geq 0$. Given such a $u_n : \mathrm{U_n} \to \mathrm{B}_x S_{\mathrm{n}}$ and $v_m : \mathrm{V_m} \to \mathrm{B}_x \mathrm{S_m}$, a morphism from $u_n \to v_m$ is a commutative square:

$$\begin{array}{ccc} \mathrm{U_n} & \xrightarrow{\alpha'} & \mathrm{V_m} \\ \downarrow{\scriptstyle u_n} & & \downarrow{\scriptstyle v_m} \\ \mathrm{B}_x S_{\mathrm{n}} & \xrightarrow{\alpha} & \mathrm{B}_x S_{\mathrm{m}} \end{array}$$

where α is a structure map of the simplicial algebraic space $\mathrm{B}_x S$. The Étale site $Et(\mathrm{B}_x S)$ is defined similarly. An abelian sheaf F on $Smt(\mathrm{B}_x S)$ is given by a collection of abelian sheaves $F = \{F_n | n\}$ with each F_n being an abelian sheaf on $Smt(\mathrm{B}_x S_{\mathrm{n}})$, so that it comes equipped with the following data: for each structure map $\alpha : \mathrm{B}_x S_{\mathrm{n}} \to \mathrm{B}_x S_{\mathrm{m}}$, one is provided with a map of sheaves $\phi_{n,m} : \alpha^*(F_m) \to F_n$ so that the maps $\{\phi_{n,m} | n, m\}$ are compatible. Abelian sheaves on the site $Et(\mathrm{B}_x S)$ may be defined similarly. We skip the verification that the category of abelian sheaves on the above sites have enough injectives. The n-th cohomology group of the simplicial object $\mathrm{B}_x S$ with respect to an abelian sheaf F is defined as the n-th right derived functor of the functor sending

$$F \mapsto kernel(\delta^0 - \delta^1 : \Gamma(\mathrm{B}_x S_0, \mathrm{F}_0) \to \Gamma(\mathrm{B}_x S_1, \mathrm{F}_1)). \tag{3.20}$$

Now we obtain the following Proposition.

Proposition 3.6. *Let F denote an abelian sheaf on $Smt(S)$. Then we obtain the following isomorphisms:*

(i) $\mathrm{H}^*_{\mathrm{smt}}(\mathrm{B}_x S, x_\bullet^*(F)) \cong \mathrm{H}^*_{\mathrm{smt}}(S, F)$, where the subscript smt denotes cohomology computed on the smooth sites and $x_\bullet : \mathrm{B}_x S \to S$ is the simplicial map induced by $x : \mathrm{X} \to S$.

(ii) $\mathrm{H}^*_{\mathrm{smt}}(\mathrm{B}_x S, x_\bullet^*(F)) \cong \mathrm{H}^*_{\mathrm{et}}(\mathrm{B}_x S, \alpha_* x_\bullet^*(F))$*, where the subscript et denotes cohomology computed on the étale site and* $\alpha : Smt(\mathrm{B}_x S) \rightarrow \mathrm{Et}(\mathrm{B}_x S)$ *is the obvious morphism of sites.*

Proof. Observe that $x : \mathrm{X} \rightarrow S$ is a covering of the stack S in the smooth topology, so that

$$kernel(\delta^0 - \delta^1 : \Gamma(\mathrm{B}_x S_0, F_0) \rightarrow \Gamma(\mathrm{B}_x S_1, F_1)) \cong \Gamma(S, F).$$

Since $\mathrm{H}^{\mathrm{n}}_{\mathrm{smt}}(S, F)$ is the n-th right derived functor of the above functor, in view of (3.20), we see that it identifies with $\mathrm{H}^{\mathrm{n}}_{\mathrm{smt}}(\mathrm{B}_x S, x_\bullet^*(F))$. This provides the isomorphism in (i). The isomorphism in (ii) is a straight-forward extension of a well-known result comparing the cohomology of an algebraic space computed on the smooth and étale sites. □

Corollary 3.7. *Assume the above context.*

(i) Then we obtain an isomorphism

$$\mathrm{H}^1_{\mathrm{et}}(\mathrm{EG}^{\mathrm{gm,m}} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^1_{\mathrm{smt}}([\mathrm{X}/\mathrm{G}], \mathbb{G}_{\mathrm{m}}), \text{ for } \mathrm{m} > 1,$$

which is functorial in the G*-scheme* X*.*

(ii) Moreover, we obtain isomorphisms:

$$\mathrm{H}^2_{\mathrm{et}}(\mathrm{EG}^{\mathrm{gm,m}} \times_{\mathrm{G}} \mathrm{X}, \mu_{\ell^{\mathrm{n}}}(1)) \cong \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mu_{\ell^{\mathrm{n}}}(1)) \cong \mathrm{H}^2_{\mathrm{smt}}([\mathrm{X}/\mathrm{G}], \mu_{\ell^{\mathrm{n}}}(1)) \text{ for } \mathrm{m} > 1.$$

which are functorial in the G*-scheme* X*, and where* $\ell \neq char(k)$*.*

Here $\mathrm{H}^1_{\mathrm{smt}}([\mathrm{X}/\mathrm{G}], \mathbb{G}_{\mathrm{m}})$ *and* $\mathrm{H}^2_{\mathrm{smt}}([\mathrm{X}/\mathrm{G}], \mu_{\ell^{\mathrm{n}}}(1))$ *denote the cohomology of the quotient stack* $[\mathrm{X}/\mathrm{G}]$ *computed on the smooth site.*

(iii) One obtains an isomorphism $\mathrm{Br}_{\mathrm{G}}(\mathrm{X})_{\ell^{\mathrm{n}}} \cong \mathrm{Br}([\mathrm{X}/\mathrm{G}])_{\ell^{\mathrm{n}}}$*, thereby proving that* $\mathrm{Br}_{\mathrm{G}}(\mathrm{X})_{\ell^{\mathrm{n}}}$ *is an invariant of the quotient stack* $[\mathrm{X}/\mathrm{G}]$*, for any prime* $\ell \neq char(k)$*.*

Proof. The first isomorphisms in both the statements (i) and (ii) are from Proposition 3.5. The second isomorphisms in (i) and (ii) follow from the isomorphism of the simplicial schemes: $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X} \cong \mathrm{cosk}_0^{[\mathrm{X}/\mathrm{G}]}(\mathrm{X})$ and Proposition 3.6. Next we consider the third statement.

Recall the long exact sequence in étale cohomology obtained from the Kummer sequence:

$$\begin{aligned} \rightarrow \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{\ell^{\mathrm{n}}} \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{\delta} \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mu_{\ell^{\mathrm{n}}}(1)) \rightarrow \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \\ \xrightarrow{\ell^n} \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \rightarrow \cdots \end{aligned} \tag{3.21}$$

Then the $cokernel(\mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{\ell^{\mathrm{n}}} \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}))$ maps to $\mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mu_{\ell^{\mathrm{n}}}(1))$, by a map induced by the boundary map δ: we will denote this map by $\bar{\delta}$. Then, in view of the isomorphisms in (i) and (ii), the Brauer group $\mathrm{Br}_{\mathrm{G}}(\mathrm{X})_{\ell^{\mathrm{n}}}$ identifies with the cokernel of the map $\bar{\delta}$.

In view of Proposition 3.6, the isomorphisms in (i) and (ii) and the long exact sequence (1.5), $\mathrm{Br}([\mathrm{X}/\mathrm{G}])_{\ell^{\mathrm{n}}}$ identifies with

$$kernel(\mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{\ell^{\mathrm{n}}} \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}})).$$

Again by Proposition 3.6, the isomorphisms in (i) and (ii) and the long-exact sequence (1.5), this identifies with

$$cokernel((\mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}})/\ell^{\mathrm{n}} \xrightarrow{\bar{\delta}} \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mu_{\ell^{\mathrm{n}}}(1))) \cong \mathrm{Br}_{\mathrm{G}}(\mathrm{X})_{\ell^{\mathrm{n}}}.$$

This proves the third assertion and hence Theorem 1.4. □

Proposition 3.8. *The equivariant Brauer groups are independent of the choice of an admissible gadget defined as in Definition 3.1.*

Proof. Let $\{(\mathrm{W_m}, \mathrm{U_m})|\mathrm{m}\}$ and $\{(\bar{\mathrm{W}}_m, \bar{\mathrm{U}}_m)|m\}$ denote two admissible gadgets for the given linear algebraic group G. Let G act on the given scheme X. Let $\mathrm{Z_m} = \mathrm{W_m} - \mathrm{U_m}$, $\bar{\mathrm{Z}}_m = \bar{\mathrm{W}}_m - \bar{\mathrm{U}}_\mathrm{m}$. Then one may observe that $\{(\widetilde{\mathrm{W}}_m = \mathrm{W_m} \times \bar{\mathrm{W}}_\mathrm{m}, \widetilde{\mathrm{U}}_\mathrm{m} = \mathrm{U_m} \times \bar{\mathrm{W}}_\mathrm{m} \cup \bar{\mathrm{W}}_\mathrm{m} \times \bar{\mathrm{U}}_\mathrm{m})|\mathrm{m}\}$ is also an admissible gadget. Moreover,

$$codim_{\widetilde{\mathrm{U}}_m \times_{\mathrm{G}} \mathrm{X}}(\widetilde{\mathrm{U}}_m \times_{\mathrm{G}} \mathrm{X} - (\mathrm{U_m} \times \bar{\mathrm{W}}_\mathrm{m}) \times_{\mathrm{G}} \mathrm{X}) = codim_{\mathrm{W_m}}(\mathrm{W_m} - \mathrm{U_m}), \text{ and} \tag{3.22}$$

$$codim_{\widetilde{\mathrm{U}}_m \times_{\mathrm{G}} \mathrm{X}}(\widetilde{\mathrm{U}}_m \times_{\mathrm{G}} \mathrm{X} - (\mathrm{W_m} \times \bar{\mathrm{U}}_\mathrm{m}) \times_{\mathrm{G}} \mathrm{X}) = codim_{\bar{\mathrm{W}}_m}(\bar{\mathrm{W}}_m - \bar{\mathrm{U}}_m). \tag{3.23}$$

Therefore, if $codim_{\mathrm{W_m}}(\mathrm{W_m} - \mathrm{U_m}) \geq 2$ and $codim_{\bar{\mathrm{W}}_m}(\bar{\mathrm{W}}_m - \bar{\mathrm{U}}_m) \geq 2$, then in the long exact sequence

$$\cdots \to \mathrm{H}^{2,1}_{\mathrm{Q_m}}(\widetilde{\mathrm{U}}_\mathrm{m} \times_{\mathrm{G}} \mathrm{X}) \to \mathrm{H}^{2,1}(\widetilde{\mathrm{U}}_\mathrm{m} \times_{\mathrm{G}} \mathrm{X}) \to \mathrm{H}^{2,1}((\mathrm{U_m} \times \bar{\mathrm{W}}_\mathrm{m}) \times_{\mathrm{G}} \mathrm{X}) \to \mathrm{H}^{3,1}_{\mathrm{Q_m}}(\widetilde{\mathrm{U}}_\mathrm{m} \times_{\mathrm{G}} \mathrm{X}) \to \cdots,$$

where $\mathrm{Q_m} = \widetilde{\mathrm{U}}_\mathrm{m} \times_{\mathrm{G}} \mathrm{X} - (\mathrm{U_m} \times \bar{\mathrm{W}}_\mathrm{m}) \times_{\mathrm{G}} \mathrm{X}$, and in the long exact sequence

$$\cdots \to \mathrm{H}^{2,1}_{\bar{\mathrm{Q}}_\mathrm{m}}(\widetilde{\mathrm{U}}_\mathrm{m} \times_{\mathrm{G}} \mathrm{X}) \to \mathrm{H}^{2,1}(\widetilde{\mathrm{U}}_\mathrm{m} \times_{\mathrm{G}} \mathrm{X}) \to \mathrm{H}^{2,1}((\mathrm{W_m} \times \bar{\mathrm{U}}_\mathrm{m}) \times_{\mathrm{G}} \mathrm{X}) \to \mathrm{H}^{3,1}_{\bar{\mathrm{Q}}_\mathrm{m}}(\widetilde{\mathrm{U}}_\mathrm{m} \times_{\mathrm{G}} \mathrm{X}) \to \cdots,$$

where $\bar{\mathrm{Q}}_m = \widetilde{\mathrm{U}}_m \times_{\mathrm{G}} \mathrm{X} - (\mathrm{W_m} \times \bar{\mathrm{U}}_\mathrm{m}) \times_{\mathrm{G}} \mathrm{X}$, both the end terms are trivial, thereby showing that the middle maps in both the long exact sequences are isomorphisms. Here $\mathrm{H}^{\mathrm{i},1}$ denotes either motivic cohomology of weight 1 with $\mathbb{Z}/\ell^n$-coefficients or étale cohomology with respect to the sheaf $\mu_{\ell^n}(1)$. The assertion on the triviality of the local motivic cohomology terms above follows from Proposition 9.4, while the corresponding assertion for étale cohomology follows from well-known cohomological semi-purity statements. □

4. Proof of Corollary 1.5

Proof of Corollary 1.5. The two statements follow readily from the discussion in section 2.3: see especially (2.7). □

5. Brauer groups of stacks of the form [X/D] where D is a smooth diagonalizable group scheme and Proof of Theorem 1.6.

The main theme of this section is the determination of the Brauer groups of quotient stacks of the form [X/D], where D is a smooth diagonalizable group scheme. We begin by observing that when D is the 1-dimensional torus $\mathbb{G}_m$, then (2.10) provides a means to compute the Brauer group of the quotient stack [X/D]. We proceed to consider some of the remaining cases.

5.1. **Toric stacks.** Let X be a toric variety defined over a field k for the split torus $\mathrm{T} = \mathbb{G}_\mathrm{m}^\mathrm{s}$. Consider a homomorphism

$$\mathbb{G}_m^r \to \mathbb{G}_m^s$$

given by characters

$$\alpha_1, \ldots, \alpha_s : \mathbb{G}_m^r \to \mathbb{G}_m.$$

Each character in turn decomposes as $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ir})$ where $\alpha_{ij} : \mathbb{G}_m \to \mathbb{G}_m$ given by $z \mapsto z^{\alpha_{ij}}$.

We are interested in the Brauer group of the quotient stack $[\mathrm{X}/\mathbb{G}_\mathrm{m}^\mathrm{r}]$: as defined in the introduction, such stacks are what we call toric stacks. We let

$$\mathrm{X}_2(\mathrm{X}, \mathbb{G}_\mathrm{m}^\mathrm{r}) := (\mathbb{A}^2 \setminus \{0\})^\mathrm{r} \times_{\mathbb{G}_\mathrm{m}^\mathrm{r}} \mathrm{X}. \tag{5.1}$$

Proposition 5.1. *Suppose that* X *is smooth. Then there is a canonical isomorphism, for any ℓ prime to char(k):*

$$\mathrm{Br}(\mathrm{X}_2(\mathrm{X}, \mathbb{G}_{\mathrm{m}}^{\mathrm{r}}))_{\ell^{\mathrm{n}}} \cong \mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^{\mathrm{r}}])_{\ell^{\mathrm{n}}}.$$

Proof. This is proved in Theorem 1.4. □

5.2. **Notation.** We let

$$\mathbb{AG}(u) = \begin{cases} \mathbb{G}_m \times \mathbb{A}^1 & \text{if } u = 1 \\ \mathbb{A}^1 \times \mathbb{G}_m & \text{if } u = 2 \end{cases}$$

Given $v \in \{1, 2\}^r$, with $v = (v_1, \ldots, v_r)$ we set

$$\mathrm{U}(v, \mathrm{X}, \mathbb{G}_{\mathrm{m}}^{\mathrm{r}}) = (\prod_1^{\mathrm{r}} \mathbb{AG}(v_i)) \times_{\mathbb{G}_{\mathrm{m}}^{\mathrm{r}}} \mathrm{X}. \tag{5.2}$$

Given $u \in \{1, 2\}$. we let

$$\hat{u} = \begin{cases} 1 & \text{if } u = 2 \\ 2 & \text{if } u = 1. \end{cases}$$

Proposition 5.2. *In the above situation, for each choice of $v = (v_1, \cdots, v_r)$, there is an isomorphism*

$$\phi_v : \mathrm{U}(v, \mathrm{X}, \mathbb{G}_{\mathrm{m}}^{\mathrm{r}}) \to \mathbb{A}^{\mathrm{r}} \times \mathrm{X}$$

given by

$$\phi_v((x_{11}, x_{12}), \ldots (x_{r1}, x_{r2}), p) = \left(\frac{x_{1\hat{v_1}}}{x_{1v_1}}, \ldots, \frac{x_{r\hat{v_r}}}{x_{rv_r}}, \prod_1^r x_{iv_i}^{-1}.p \right)$$

Proof. One checks that the lift of ϕ_v, defined by the same equations is equivariant for the action of $\mathbb{G}_m^r$ on $\prod_1^r \mathbb{A}^{v_i} \times \mathrm{X}$ and hence ϕ_v is well defined. One can write down the inverse of ϕ_v as

$$\phi_v^{-1}(x_1, \ldots x_r, p) = [(y_{11}, y_{12}), \ldots (y_{r1}, y_{r2}), p]$$

where

$$y_{i1} = \begin{cases} 1 & \text{if } v_i = 1 \\ x_i & \text{if } v_i = 2 \end{cases}$$

and

$$y_{i2} = \begin{cases} x_i & \text{if } v_i = 1 \\ 1 & \text{if } v_i = 2 \end{cases} .$$

One checks that these maps are mutually inverse. □

Proposition 5.3. *As v varies over $\{1, 2\}^r$ the* $\mathrm{U}(v, \mathrm{X}, \mathbb{G}_{\mathrm{m}}^{\mathrm{r}})$ *form an open cover of* $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_{\mathrm{m}}^{\mathrm{r}})$.

Proof. This is clear. □

Proposition 5.4. *The variety* $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_{\mathrm{m}}^{\mathrm{r}})$ *is toric and has dense open torus*

$$\mathbb{G}_m^r \times \mathrm{T} \xrightarrow{\lambda} \mathrm{X}_2(\mathrm{X}, \mathbb{G}_{\mathrm{m}}^{\mathrm{r}})$$

given by

$$(z_1, \ldots, z_r, p) \longmapsto [(1, z_1), (1, z_2), \ldots, (1, z_r), p].$$

Proof. The question that λ is an open embedding is local on $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_\mathrm{m}^\mathrm{r})$ so it can be checked locally on $\mathrm{U}(v, \mathrm{X}, \mathbb{G}_\mathrm{m}^\mathrm{r})$. Using the isomorphism of the prior proposition, we check that it is an open embedding with dense image.

To see that $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_\mathrm{m}^\mathrm{r})$ is normal note that normality is local for the smooth topology and the smooth cover $\mathrm{X} \times (\mathbb{A} - \{0\})^\mathrm{r}$ is normal. □

Proposition 5.5. *Choose $v, w \in \{1, 2\}^r$.*

(1) We have $\phi_v(\mathrm{U}(v, \mathrm{X}, \mathbb{G}_\mathrm{m}^\mathrm{r}) \cap \mathrm{U}(w, \mathrm{X}, \mathbb{G}_\mathrm{m}^\mathrm{r})) = \prod_1^\mathrm{r} \mathbb{A}\mathbb{G}^{v_i, w_i} \times \mathrm{X}$ where

$$\mathbb{A}\mathbb{G}^{v_i, w_i} = \begin{cases} \mathbb{A}^1 & \text{if } v_i = w_i \\ \mathbb{G}_m & \text{otherwise.} \end{cases}$$

(2) The composition

$$\prod_1^r \mathbb{A}\mathbb{G}^{v_i, w_i} \times X \overset{\phi_v^{-1}}{\to} \mathrm{U}(v, \mathrm{X}, \mathbb{G}_\mathrm{m}^\mathrm{r}) \cap \mathrm{U}(w, \mathrm{X}, \mathbb{G}_\mathrm{m}^\mathrm{r}) \overset{\phi_w}{\to} \prod_1^\mathrm{r} \mathbb{A}\mathbb{G}^{v_i, w_i} \times \mathrm{X}$$

is given by

$$\phi_w \circ \phi_v^{-1}(x_1, \ldots, x_r, p) = (x_1^{v_1, w_1}, \ldots, x_r^{v_r, w_r}, t \cdot p)$$

where

$$x_i^{v_i, w_i} = \begin{cases} x_i & \text{if } v_i = w_i \\ x_i^{-1} & \text{otherwise,} \end{cases}$$

and

$$t = \prod_{i=1, v_i \neq w_i}^r x_i^{-1}.$$

Proof. (1) Set

$$\widetilde{\mathbb{A}\mathbb{G}^{v_i, w_i}} = \begin{cases} \mathbb{G}_m \times \mathbb{A}^1 & \text{if } v_i = w_i = 1 \\ \mathbb{G}_m \times \mathbb{G}_m & \text{if } v_i \neq w_i \\ \mathbb{A}^1 \times \mathbb{G}_m & \text{if } v_i = w_i = 2. \end{cases}$$

It follow from 5.2 that $\prod_1^r \widetilde{\mathbb{A}^{v_i, w_i}} \times_{\mathbb{G}_m^r} \mathrm{X} = \mathrm{U}(v, \mathrm{X}, \mathbb{G}_\mathrm{m}^\mathrm{r}) \cap \mathrm{U}(w, \mathrm{X}, \mathbb{G}_\mathrm{m}^\mathrm{r})$.

When $v_i = w_i$ the ith component of $\widetilde{\mathbb{A}^{v_i, w_i}}$ is the same as that of $U(v, X, \mathbb{G}_m^r)$ and when $v_i \neq w_i$ then an open axis is missing. The result follows directly from the formula for ϕ_v

(2) We make use of the formula for ϕ_v^{-1} given in the proof of 5.2. We see that

$$\begin{aligned} \phi_w \circ \phi_v^{-1}(x_1, \ldots, x_r, p) &= \phi_w((y_{11}, y_{12}), \ldots, (y_{r1}, y_{r2}), p) \\ &= \left(\frac{y_{1, \hat{w}_1}}{y_{1 w_1}}, \ldots, \frac{y_{r, \hat{w}_r}}{y_{r w_r}}, \prod_1^r y_{i w_i}^{-1} \cdot p \right), \end{aligned}$$

where

$$y_{i1} = \begin{cases} 1 & \text{if } v_i = 1 \\ x_i & \text{if } v_i = 2 \end{cases} \qquad y_{i2} = \begin{cases} x_i & \text{if } v_i = 1 \\ 1 & \text{if } v_i = 2. \end{cases}$$

One checks

$$y_{i w_i} = \begin{cases} x_i & \text{if } v_i = w_i \\ 1 & \text{if } v_i \neq w_i. \end{cases} \qquad y_{i \hat{w}_i} = \begin{cases} 1 & \text{if } v_i = w_i \\ x_i & \text{if } v_i \neq w_i. \end{cases}$$

It follows that

$$\frac{y_{i \hat{w}_i}}{y_{i w_i}} = \begin{cases} x_i & \text{if } v_i = w_i \\ x_i^{-1} & \text{if } v_i \neq w_i \end{cases}$$

and

$$\prod_1^r y_{i,w_i}^{-1} = \prod_{v_i \neq w_i} x_i^{-1} = t.$$

□

We will assume from now on familiarity with cones and fans and their associated toric varieties.

Proposition 5.6. *Consider the morphism of tori*

$$\phi_v : \mathrm{U}(v, \mathrm{X}, \mathbb{G}_{\mathrm{m}}^{\mathrm{r}}) \supseteq \mathbb{G}_{\mathrm{m}}^{\mathrm{r}} \times \mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\mathrm{r+s}} \to \mathbb{G}_{\mathrm{m}}^{\mathrm{r+s}} \subseteq \mathbb{A}^{\mathrm{r}} \times \mathrm{X}.$$

Then the induced pullback map on character lattices $\phi_v^* : \mathbb{X}(\mathbb{G}_m^{r+s}) = \mathbb{Z}^{r+s} \to \mathbb{X}(\mathbb{G}_m^{r+s}) = \mathbb{Z}^{r+s}$ *is given by multiplication by the matrix*

$$B(v, \alpha) = \begin{pmatrix} I_v & A(v,\alpha) \\ 0_{s\times r} & I_{s\times s} \end{pmatrix},$$

where:

$$I_v = \begin{pmatrix} \epsilon(v_1) & 0 & \cdots & 0 \\ 0 & \epsilon(v_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \epsilon(v_r) \end{pmatrix}$$

$$A(v, \alpha) = \begin{pmatrix} -\alpha_{11}(v_1 - 1) & \cdots & -\alpha_{s1}(v_1 - 1) \\ -\alpha_{12}(v_2 - 1) & \cdots & -\alpha_{s2}(v_2 - 1) \\ \vdots & \vdots & \vdots \\ -\alpha_{1r}(v_r - 1) & \cdots & -\alpha_{sr}(v_r - 1) \end{pmatrix}$$

Here

$$\epsilon(v_i) = \begin{cases} 1 & \text{if } v_i = 1 \\ -1 & \text{if } v_i = 2 \end{cases}$$

Proof. We need to compute the morphism from the torus $\mathbb{G}_m^r \times \mathrm{T}$ to $\mathbb{A}^r \times \mathrm{X}$ via ϕ_v. The torus $\mathbb{G}_m^r \times \mathrm{T}$ sits inside $\mathrm{U}(v, \mathrm{X}, \mathbb{G}_{\mathrm{m}}^{\mathrm{r}})$ as described in 5.4. Recall the embedding is

$$(z_1, \ldots, z_r, p) \longmapsto [(1, z_1), \ldots, (1, z_r), p].$$

Hence, one computes

$$\phi_v((1, z_1), (1, z_2), \ldots, (1, z_r), p_1, \ldots, p_s) = \left(z_1^{\epsilon(v_1)}, z_2^{\epsilon(v_2)}, \ldots, z_r^{\epsilon(v_r)}, tp_1, tp_2, \ldots, tp_s\right)$$

where

$$t = \prod_{i=1, v_i=2}^r z_i^{-1}.$$

Now let $\chi_i : \mathbb{G}_m^{r+s} \to \mathbb{G}_m$ be the character $\chi_i(z_1, \ldots, z_{r+s}) = z_i$. We need to compute $\chi_i \circ \phi_v$. It is easy to see that

$$\chi_i \circ \phi_v = \chi_i^{\epsilon(v_i)}$$

when $i \leq r$. This proves that the left $(r+s) \times r$ block of the matrix $B(v, \alpha)$ is correct. Now assume $i > r$. Take $1 \leq j \leq r$. Then

$$\chi_i \circ \phi_v([(1,1), \ldots, (1, z_j), \ldots, (1,1), 1]) = \chi_i(1, \ldots, z_j^{\epsilon(v_j)}, \ldots, 1, z_j^{\lambda}.1).$$

Now $\lambda = 0$ if $v_j = 1$ and $\lambda = -1$ if $v_j = 2$. Now recall that the torus $\mathbb{G}_m^r$ acts on $T = \mathbb{G}_m^s$ via the characters α_i as described at the start of this section. The top right $r \times s$ block of $B(v, \alpha)$ is obtained by observing that

$$v_i - 1 = \begin{cases} 0 & \text{if } v_i = 1 \\ 1 & \text{if } v_i = 2. \end{cases}$$

It remains to check that the bottom $s \times s$ block is the identity. This straightforward as $\mathbb{G}_m^s$ act on $\mathbb{G}_m^s$ via ordinary multiplication. □

Proposition 5.7. *Let* V *be a finite dimensional vector space with an automorphism* $a : \mathrm{V} \to \mathrm{V}$*. Let* ρ *be a cone in* V *with dual cone* $\rho^\vee$*. Then we have*

$$a(\rho)^\vee = (a^t)^{-1}(\rho^\vee).$$

Proof.

$$\begin{aligned} v \in (a(\rho))^\vee &\iff < a(x), v > 0 \geq \quad \forall\, x \in \rho \\ &\iff < x, a^t(v) > 0 \geq \quad \forall\, x \in \rho \\ &\iff (a^t)(v) \in \rho^\vee \\ &\iff v \in (a^t)^{-1}(\rho^\vee). \end{aligned}$$

□

Proposition 5.8. *Let* $\Sigma \subseteq \mathbb{Z}^s$ *be a fan for the toric variety* X*. Let* $\mathrm{cone}(\mathrm{e}_1, \ldots, \mathrm{e}_\mathrm{r})$ *be the standard fan for the toric variety* $\mathbb{A}^r$ *so that* $\mathrm{cone}(\mathrm{e}_1, \ldots, \mathrm{e}_\mathrm{r}) \times \Sigma$ *is a fan for* $\mathbb{A}^r \times \mathrm{X}$*. Then a fan for* $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_\mathrm{m}^\mathrm{r})$ *is given by taking the union of the fans*

$$(B(v, \alpha)^t)^{-1}(\mathrm{cone}(\mathrm{e}_1, \ldots, \mathrm{e}_\mathrm{r}) \times \Sigma)$$

as v *varies over* $\{1, 2\}^r$*.*

Proof. Consider first the case where X is affine, given by a cone σ. We write S_σ for the monoid consisting of lattice points in $\sigma^\vee$. We have

$$\mathrm{Spec}(k[S_{\sigma \times \mathrm{cone}(\mathrm{e}_1 \ldots \mathrm{e}_\mathrm{r})}]) = \mathrm{X}.$$

As a sub-algebra of the co-ordinate ring of the torus, this is

$$k[S_{B(v,\alpha)(\sigma \times \mathrm{cone}(\mathrm{e}_1 \ldots \mathrm{e}_\mathrm{r}))}].$$

In other words, the dual cone for $\mathrm{U}(v, \mathrm{X}, \mathbb{G}_\mathrm{m}^\mathrm{r})$ is $\mathrm{B}(v, \alpha)(\sigma \times \mathrm{cone}(\mathrm{e}_1 \ldots \mathrm{e}_\mathrm{r}))$. The cone is then described via the lemma. The affine case now follows from [CLS, Exercise 3.2.11]. The same gluing procedure yields the result in general. □

We will now recall the following basic framework from [JL, section 2]. Observe that $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_\mathrm{m}^\mathrm{r})$ is now a smooth split toric variety for the open torus $\mathbb{G}_m^{r+s}$. We will denote the characters of this torus by t_i, $i = 1, \cdots r + s$. Let ζ denote a primitive ℓ^n-th root of unity in k and let $(t_i, t_j)_\zeta$ and $(b, t_i)_\zeta$ denote cyclic algebras with $b \in k^*$. Observe that any Azumaya algebra generated by the cyclic algebras $(t_i, t_j)_\zeta$, $i < j$ will be of the form $\Pi_{i<j}(t_i, t_j)_\zeta^{e_{i,j}}$, for some choice of integers $0 \leq e_{i,j} < \ell^n$, while any Azumaya algebra generated by the cyclic algebras $(b, t)_\zeta$, with $b \in k^*$ and t a character of the torus $\mathbb{G}_m^{r+s}$ will be of the form $\Lambda = \Pi_{i=1}^r (b_i, t_i)_\zeta^{e_i}$, for some integers $e_i \geq 0$.

We will denote the subgroup of $\mathrm{Br}(\mathrm{T})_{\ell^{\mathrm{n}}}$ generated by $\{\Pi_{1\le i<j\le r+s}(t_i,t_j)_\zeta^{e_{i,j}}|e_{i,j}\ge 0\}$ by A, and the subgroup generated by $\{\Pi_{1\le i\le r+s}(b,t_i)_\zeta^{e_i}|e_i\ge 0, b\in k^*\}$ by B. Let M (N) denote the lattice of characters (co-characters or 1-parameter subgroups) associated to the split torus $\mathbb{G}_m^{r+s}$. Let Δ denote the fan associated to $\mathrm{X}_2(\mathrm{X},\mathbb{G}_{\mathrm{m}}^{\mathrm{r}})$ and let N′ denote the subgroup generated by $\bigcup_{\sigma\in\Delta}\sigma\cap\mathrm{N}$. Then $\mathrm{N}'=\mathbb{Z}a_1\mathrm{n}_1\oplus\cdots\mathbb{Z}a_{r+s}\mathrm{n}_{\mathrm{r+s}}$, where $\mathrm{n}_1,\cdots,\mathrm{n}_{r+s}$ is a basis for N and $a_i\ge 0$ are integers with $a_i|a_{i+1}$, for $i=1,\cdots,r+s-1$. Then [JL, Theorem 2.1] readily provides the following theorem that calculates the Brauer group of the toric stack $[\mathrm{X}/\mathbb{G}_{\mathrm{m}}^{\mathrm{r}}]$.

Theorem 5.9. *For a prime ℓ different from char(k), let $\mathrm{Br}(\mathrm{X})\{\ell\}$ denote the ℓ-primary torsion in the Brauer group of* X. *Let n denote a fixed positive integer. Then the following hold assuming k contains a primitive ℓ^n-th root of unity ζ:*

(i) $\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^{\mathrm{r}}])_{\ell^{\mathrm{n}}}\cap\mathrm{A}=\mathrm{Br}(\mathrm{X}_2(\mathrm{X},\mathbb{G}_{\mathrm{m}}^{\mathrm{r}})_{\ell^{\mathrm{n}}}\cap\mathrm{A}$ *is generated by* $\{\Lambda=\Pi_{i<j}(t_i,t_j)_\zeta^{e_{i,j}}|\ \ell^n|a_i, i=1,\cdots r+s\}$, *and*

(ii) $\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^{\mathrm{r}}])_{\ell^{\mathrm{n}}}\cap\mathrm{B}=\mathrm{Br}(\mathrm{X}_2(\mathrm{X},\mathbb{G}_{\mathrm{m}}^{\mathrm{r}})_{\ell^{\mathrm{n}}}\cap\mathrm{B}$ *is generated by* $\{\Lambda(b)=\Pi_{i=1}^r(b,t_i)_\zeta^{e_i}|\ \ell^n|a_i, i=1,\cdots,r+s\}$ *as $b\in k^*$ varies among the corresponding classes in* $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,k,\mu_{\ell^{\mathrm{n}}}(0))$.

(iii) *Moreover,* $\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^{\mathrm{r}}])_{\ell^{\mathrm{n}}}\cong\mathrm{Br}(\mathrm{Spec}\,k)_{\ell^{\mathrm{n}}}\oplus(\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^{\mathrm{r}}])_{\ell^{\mathrm{n}}}\cap\mathrm{A})\oplus(\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^{\mathrm{r}}])_{\ell^{\mathrm{n}}}\cap\mathrm{B})$.

Moreover, [JL, Corollary 2.2] provides the following corollary.

Corollary 5.10. *Assume the basic hypotheses of the last theorem. Then the following hold:*

(i) *In case there is a cone σ in the fan Δ with dimension$(\sigma)\ge r-1$, then* $\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^{\mathrm{r}}])_{\ell^{\mathrm{n}}}\cap\mathrm{A}$ *is trivial.*

(ii) *In case there is a cone σ in the fan Δ with dimension$(\sigma)\ge r$, then both* $\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^{\mathrm{r}}])_{\ell^{\mathrm{n}}}\cap\mathrm{B}$ *and* $\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^{\mathrm{r}}])_{\ell^{\mathrm{n}}}\cap\mathrm{A}$ *are trivial, so that* $\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^{\mathrm{r}}])_{\ell^{\mathrm{n}}}\cong\mathrm{Br}(\mathrm{Spec}\,k)_{\ell^{\mathrm{n}}}$.

Example 5.11. *Let $n\ge 1$ denote a positive integer. We will now consider the Brauer group of the weighted projective stack $[(\mathbb{A}^n-\{0\})/\mathbb{G}_m]$.*

Corollary 5.12. *Assume the basic hypotheses of Theorem 5.9. Then* $\mathrm{Br}([(\mathbb{A}^{\mathrm{n}}-\{0\})/\mathbb{G}_{\mathrm{m}}])_{\ell^{\mathrm{n}}}\cong\mathrm{Br}(\mathrm{Spec}\,k)_{\ell^{\mathrm{n}}}$.

Proof. From the description of the fan for the associated toric variety $\mathrm{X}_2(\mathbb{A}^{\mathrm{n}}-\{0\},\mathbb{G}_{\mathrm{m}})$ as in Proposition 5.8, one can see that it satisfies hypotheses in Corollary 5.10(ii). □

5.3. **Quotient stacks of the form** $[\mathrm{X}/\mu_{\ell^{\mathrm{n}}}]$**.** Next we proceed to consider quotient stacks of the form $[\mathrm{X}/\mu_{\ell^{\mathrm{n}}}]$ where X is a smooth scheme provided with an action by the diagonalizable group scheme of the form μ_{ℓ^n}, all defined over the base scheme B as in 1.1, with ℓ a prime invertible in $\mathcal{O}_{\mathrm{B}}$. In this case, we let $\mathrm{E}\mu_{\ell^n}=\mathbb{A}^2-\{0\}$ provided with the action of μ_{ℓ^n}, where it acts through the obvious injection $\mu_{\ell^n}\to\mathbb{G}_m$ and $\mathbb{G}_m$ acts on $\mathbb{A}^2$ in the obvious manner. In view of Theorem 1.4, one may identify $\mathrm{Br}([\mathrm{X}/\mu_{\ell^{\mathrm{n}}}])$ with $\mathrm{Br}(\mathrm{E}\mu_{\ell^{\mathrm{n}}}\times_{\mu_{\ell^{\mathrm{n}}}}\mathrm{X})$.

Proposition 5.13. *Assume the above situation. Then the following hold.*

(i) For $n\ge n'$, we obtain a short exact sequence:

$$0\to\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}])_{\ell^{\mathrm{n}'}}\to\mathrm{Br}([\mathrm{X}/\mu_{\ell^{\mathrm{n}}}])_{\ell^{\mathrm{n}'}}\to\mathrm{H}^1_{\mathrm{et}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}],\mu_{\ell^{\mathrm{n}'}}(0))\to 0.$$

(ii) In case $n<m$, we obtain a short exact sequence, where σ is the first Chern classes of the line bundle $\mathcal{O}(-1)$ on $\mathrm{B}\mathbb{G}_{\mathrm{m}}=\mathbb{P}^\infty$:

$$0\to(\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}})_{\ell^{\mathrm{n}'}})/\ell^{\mathrm{n}}\sigma\to\mathrm{Br}([\mathrm{X}/\mu_{\ell^{\mathrm{n}}}])_{\ell^{\mathrm{n}'}}\to\ker(\mathrm{H}^1_{\mathrm{et}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}],\mu_{\ell^{\mathrm{n}'}}[0])\xrightarrow{\ell^{\mathrm{n}}\bar{\sigma}}\mathrm{H}^{3,1}_{\mathrm{et}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}]))\to 0.$$

(iii) Next assume that the base field has a primitive ℓ-th root of unity and that the smooth scheme X is provided with an action by the symmetric group Σ_ℓ, where Σ_ℓ denotes the symmetric group on ℓ-letters. Then μ_ℓ identifies with the constant subsheaf $\mathbb{Z}/\ell$ of Σ_ℓ and

$$\mathrm{Br}([\mathrm{X}/\Sigma_\ell])_\ell \cong (\mathrm{Br}([\mathrm{X}/\mu_\ell])_\ell)^{\mathrm{Aut}(\mu_\ell)}.$$

Proof. We will first prove (i) and (ii) when X = B, so that μ_{ℓ^n} acts trivially. We begin with the calculations in [Voev2, section 6] on the motivic cohomology of $\mathrm{B}\mu_{\ell^n}$. A key observation is that $\mathrm{B}\mu_{\ell^n} = \mathcal{O}(-\ell^n) - \mathrm{z}(\mathbb{P}^\infty)$, where $\mathcal{O}(-\ell^n)$ denotes the obvious line bundle on $\mathbb{P}^\infty$. This is clear since a model for the geometric classifying space for μ_{ℓ^n} is given as the quotient $(\mathbb{A}^{n+1}-0)/\mu_{\ell^n}$ which fibers over $\mathbb{P}^n = (\mathbb{A}^{n+1}-0)/\mathbb{G}_m$. Therefore, the homotopy purity theorem [MV, Theorem 2.23] provides the cofiber sequence:

$$\mathrm{B}\mu_{\ell^n +} \rightarrow (\mathcal{O}(-\ell^n)_{\mathbb{P}^\infty})_+ \rightarrow \mathrm{Th}(\mathcal{O}(-\ell^n)), \tag{5.3}$$

where $\mathrm{Th}(\mathcal{O}(-\ell^n))$ is the Thom-space of the above line bundle, which identifies with the cofiber of the first map. Next we let $\mathrm{H}^{\mathrm{even}}_{\mathrm{M}}(\mathbb{P}^\infty, \mathbb{Z}/\ell^{n'}) = \oplus_i \mathrm{H}^{2i,i}_{\mathrm{M}}(\mathbb{P}^n, \mathbb{Z}/\ell^{n'})$ and similarly $\mathrm{H}^{\mathrm{even}}_{\mathrm{et}}(\mathbb{P}^\infty, \mathbb{Z}/\ell^{n'}) = \oplus_i \mathrm{H}^{2i,i}_{\mathrm{et}}(\mathbb{P}^n, \mathbb{Z}/\ell^{n'})$. Then

$$\begin{aligned} &\mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathbb{P}^\infty, \mathbb{Z}/\ell^{n'}) \cong \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{B}, \mathbb{Z}/\ell^{n'}) \underset{\mathbb{Z}/\ell^{n'}}{\otimes} \mathrm{H}^{\mathrm{even}}_{\mathrm{M}}(\mathbb{P}^\infty, \mathbb{Z}/\ell^{n'}), \text{ and} \\ &\mathrm{H}^{*,\bullet}_{\mathrm{et}}(\mathbb{P}^\infty, \mathbb{Z}/\ell^{n'}) \cong \mathrm{H}^{*,\bullet}_{\mathrm{et}}(\mathrm{B}, \mathbb{Z}/\ell^{n'}) \underset{\mathbb{Z}/\ell^{n'}}{\otimes} \mathrm{H}^{\mathrm{even}}_{\mathrm{et}}(\mathbb{P}^\infty, \mathbb{Z}/\ell^{n'}). \end{aligned} \tag{5.4}$$

Moreover, $\mathrm{H}^{\mathrm{even}}_{\mathrm{M}}(\mathbb{P}^\infty, \mathbb{Z}/\ell^{n'}) = \mathbb{Z}/\ell^{n'}[\sigma]$ and $\mathrm{H}^{\mathrm{even}}_{\mathrm{et}}(\mathbb{P}^\infty, \mathbb{Z}/\ell^{n'}) = \mathbb{Z}/\ell^{n'}[\bar{\sigma}]$ are both polynomial rings in the variable σ and $\bar{\sigma}$ which are the first Chern classes of the line bundle $\mathcal{O}(-1)$. (Observe that the cycle map sends σ to $\bar{\sigma}$.)

Then the long-exact sequences in motivic and in étale cohomology with $\mathbb{Z}/\ell^{n'}$ coefficients associated to the above cofiber sequence provide the commutative diagram of long exact sequences

(5.5)

$$\begin{array}{ccccccccc} \rightarrow & \mathrm{H}^{0,0}_{\mathrm{M}}(\mathrm{B}, \mathbb{Z}/\ell^{n'})[\sigma] & \xrightarrow{\cup e} & \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{B}, \mathbb{Z}/\ell^{n'})[\sigma] & \rightarrow & \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{B}\mu_{\ell^n}, \mathbb{Z}/\ell^{n'}) & \rightarrow & \mathrm{H}^{1,0}_{\mathrm{M}}(\mathrm{B}, \mathbb{Z}/\ell^{n'})[\sigma] & \rightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \rightarrow & \mathrm{H}^{0}_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^{n'}}(0))[\bar{\sigma}] & \xrightarrow{\cup e} & \mathrm{H}^{2,1}_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^m}(1))[\bar{\sigma}] & \rightarrow & \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{B}\mu_{\ell^n}, \mu_{\ell^{n'}}) & \rightarrow & \mathrm{H}^{1}_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^{n'}}(0))[\bar{\sigma}] & \rightarrow \end{array}$$

where $\sigma \epsilon \mathrm{H}^{2,1}(\mathbb{P}^\infty)$ is the first Chern class of $\mathcal{O}(-1)$.

Here e denotes the Euler class of the line bundle $\mathcal{O}(-\ell^n)$. $\mathrm{H}^{*,\bullet}$ denotes either motivic or étale cohomology with $\mathbb{Z}/\ell^m$-coefficients. The map denoted $\cup e$ is the composition of the Thom isomorphism and the obvious map $\mathrm{H}^{*,\bullet}(\mathrm{Th}(\mathcal{O}(-\ell^n))) \rightarrow \mathrm{H}^{*,\bullet}(\mathrm{E}(\mathcal{O}(-\ell^n)_{\mathbb{P}^\infty})) \cong \mathrm{H}^{*,\bullet}(\mathbb{P}^\infty)$.

Next we break the remaining part of the proof into two cases, (i) when $n \geq n'$ and (ii) when $n < n'$. In the first case, observe that $\ell^n \sigma = e(\mathcal{O}(-\ell^n))$ and therefore, the above long exact sequence breaks up into short exact sequences since we are working with $\mathbb{Z}/\ell^{n'}$-coefficients for $n \geq n'$. In view of (5.4), the statement in (i) follows when $n \geq n'$ by taking the cokernels of the vertical maps, and observing that $\mathrm{H}^{1,0}_{\mathrm{M}} = 0$. In fact one may observe that the vertical maps above are all injective, as one may see from the Kummer sequence and then apply Lemma 2.1 to obtain the required conclusion, when X = B and the action by μ_{ℓ^n} is trivial.

Next we consider the case where X is no longer the base scheme B. In this case, and for the remainder of the proof, we will denote $(\mathbb{A}^2 - \{0\}) \times_{\mu_{\ell^n}} \mathrm{X}$ by $[\mathrm{X}/\mu_{\ell^{\mathrm{n}}}]$ and similarly $(\mathbb{A}^2 - \{0\}) \times_{\mathbb{G}_m} \mathrm{X}$ by $[\mathrm{X}/\mathbb{G}_{\mathrm{m}}]$. A key observation now is that $(\mathbb{A}^2 - \{0\}) \times_{\mu_{\ell^n}} \mathrm{X} = \pi^*(\mathcal{O}(-\ell^{\mathrm{n}})) - \mathrm{z}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}])$, where $\pi : (\mathbb{A}^2 - \{0\}) \times_{\mu_{\ell^n}} \mathrm{X} \to \mathrm{B}\mu_{\ell^{\mathrm{n}}}$ is the projection and $\pi^*(\mathcal{O}(-\ell^n))$ denotes the pull-back of the line bundle $\mathcal{O}(-\ell^n)$ on $\mathbb{P}^\infty = \mathrm{B}\mathbb{G}_{\mathrm{m}}$. This is clear since a model for the geometric classifying space for μ_{ℓ^n} is given as the quotient $(\mathbb{A}^{n+1} - 0)/\mu_{\ell^n}$ which fibers over $\mathbb{P}^n = (\mathbb{A}^{n+1} - 0)/\mathbb{G}_m$. Therefore, the homotopy purity theorem [MV, Theorem 2.23] provides the cofiber sequence:

$$((\mathbb{A}^2 - \{0\}) \times_{\mu_{\ell^n}} \mathrm{X})_+ \to \pi^*(\mathcal{O}(-\ell^{\mathrm{n}})_{\mathbb{P}^\infty})_+ \to \mathrm{Th}(\pi^*(\mathcal{O}(-\ell^{\mathrm{n}})))). \tag{5.6}$$

In this case, in place of the diagram (5.5), we obtain the diagram:
(5.7)

$$\begin{array}{ccccccccc}
\to & \mathrm{H}^{0,0}_{\mathrm{M}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mathbb{Z}/\ell^{\mathrm{n}'}) & \xrightarrow{\cup\pi^*(e)} & \mathrm{H}^{2,1}_{\mathrm{M}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mathbb{Z}/\ell^{\mathrm{n}'}) & \to & \mathrm{H}^{2,1}_{\mathrm{M}}([\mathrm{X}/\mu_{\ell^{\mathrm{n}}}], \mathbb{Z}/\ell^{\mathrm{n}'}) & \to & \mathrm{H}^{1,0}_{\mathrm{M}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mathbb{Z}/\ell^{\mathrm{n}'}) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & \mathrm{H}^{0}_{\mathrm{et}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mu_{\ell^{\mathrm{n}'}}(0)) & \xrightarrow{\cup\pi^*(e)} & \mathrm{H}^{2}_{\mathrm{et}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mu_{\ell^{\mathrm{n}'}}(1)) & \to & \mathrm{H}^{2}_{\mathrm{et}}([\mathrm{X}/\mu_{\ell^{\mathrm{n}}}], \mu_{\ell^{\mathrm{n}'}}(1)) & \to & \mathrm{H}^{1}_{\mathrm{et}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mu_{\ell^{\mathrm{n}'}}(0)) & \to
\end{array}$$

where the map denoted $\cup\pi^*(e)$ is the cup product with the Euler class of the pulled-back line bundle $\pi^*(\mathcal{O}(-\ell^n)_{\mathbb{P}^\infty})$. Now the class $\pi^*(e) = \pi^*(\ell^n\sigma) = \ell^n\pi^*(\sigma)$ and hence is trivial. Therefore, the long-exact sequences in each row break up into short-exact sequences. One obtains the short exact sequence in (i) on taking the cokernels.

Next we consider (ii), i.e. the case when $n < n'$. Now the long exact sequence (5.5) no longer breaks up into short exact sequences so that we will argue a bit differently as follows. Here also we first consider the special case where X = B and μ_{ℓ^n} acts trivially. Then one obtains the following commutative diagram of short exact sequences from the long exact sequence (5.5):

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{B})[\sigma]/(\ell^{\mathrm{n}}\sigma) & \to & \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{B}\mu_{\ell^{\mathrm{n}}}) & \to & \mathrm{K} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}'}})[\bar{\sigma}]/(\ell^{\mathrm{n}}\bar{\sigma}) & \to & \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{B}\mu_{\ell^{\mathrm{n}}}, \mu_{\ell^{\mathrm{n}'}}) & \to & \bar{\mathrm{K}} & \to & 0
\end{array} \tag{5.8}$$

where

$$\mathrm{K} = \ker(\mathrm{H}^{1,0}_{\mathrm{M}}(\mathrm{B})[\sigma] \xrightarrow{\ell^{\mathrm{n}}\sigma} \mathrm{H}^{*+1,\bullet}_{\mathrm{M}}(\mathrm{S})[\sigma]) \text{ and } \bar{\mathrm{K}} = \ker(\mathrm{H}^{1}_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}'}}(0))[\bar{\sigma}] \xrightarrow{\ell^{\mathrm{n}}\bar{\sigma}} \mathrm{H}^{3}_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}'}})[\bar{\sigma}]).$$

Taking the cokernels of the vertical maps, now proves (ii) in this case. When X is no longer B, we will instead obtain the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{H}^{2,1}_{\mathrm{M}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}])/(\ell^{\mathrm{n}}\sigma) & \to & \mathrm{H}^{2,1}_{\mathrm{M}}([\mathrm{X}/\mu_{\ell^{\mathrm{n}}}]) & \to & \mathrm{K} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathrm{H}^{2}_{\mathrm{et}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mu_{\ell^{\mathrm{n}'}})/(\ell^{\mathrm{n}}\bar{\sigma}) & \to & \mathrm{H}^{2}_{\mathrm{et}}([\mathrm{X}/\mu_{\ell^{\mathrm{n}}}], \mu_{\ell^{\mathrm{n}'}}) & \to & \bar{\mathrm{K}} & \to & 0
\end{array} \tag{5.9}$$

where

$$\mathrm{K} = \ker(\mathrm{H}^{1,0}_{\mathrm{M}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}]) \xrightarrow{\ell^{\mathrm{n}}\sigma} \mathrm{H}^{*+1,\bullet}_{\mathrm{M}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}])) \text{ and } \bar{\mathrm{K}} = \ker(\mathrm{H}^{1}_{\mathrm{et}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mu_{\ell^{\mathrm{n}'}}(0)) \xrightarrow{\ell^{\mathrm{n}}\bar{\sigma}} \mathrm{H}^{3}_{\mathrm{et}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mu_{\ell^{\mathrm{n}'}})).$$

Taking the cokernels of the vertical maps, now proves (ii) in this case.

Next we will consider (iii). First observe that, under our hypotheses, one may identify the sheaf μ_ℓ with the constant sheaf $\mathbb{Z}/\ell$. In this case, we will adopt the terminology from section 2.3 and let $\mathrm{EGL}_n^{gm,2}$ denote the object defined in (2.6). Then we will let $[\mathrm{X}/\Sigma_\ell]$ denote $\mathrm{EGL}_n^{gm,2} \times_{\mathrm{GL}_n} (\mathrm{GL}_n \times_{\Sigma_\ell} \mathrm{X})$. Moreover we will let $[\mathrm{X}/\mu_{\ell^n}]$ denote $\mathrm{EGL}_n^{gm,2} \times_{\mathrm{GL}_n} (\mathrm{GL}_n \times_{\mathbb{Z}/\ell} \mathrm{X})$. Then one has a natural map $p : [\mathrm{X}/\mu_\ell] \overset{\cong}{\to} [\mathrm{X}/\mathbb{Z}/\ell] \to [\mathrm{X}/\Sigma_\ell]$. The main observation now is that $|\Sigma_l/\mathbb{Z}/\ell| = (\ell-1)!$ is invertible in $\mathbb{Z}/\ell$ and μ_ℓ. Therefore, the induced maps

$$(5.10)\qquad \begin{aligned} p^* &: \mathrm{H}_{\mathrm{M}}^{*,\bullet}([\mathrm{X}/\Sigma_\ell], \mathbb{Z}/\ell) \to \mathrm{H}_{\mathrm{M}}^{*,\bullet}([\mathrm{X}/\mu_\ell], \mathbb{Z}/\ell)^{\mathrm{Aut}(\mu_\ell)} \text{ and} \\ p^* &: \mathrm{H}_{\mathrm{et}}^{*}([\mathrm{X}/\Sigma_\ell], \mathbb{Z}/\ell) \to \mathrm{H}_{\mathrm{et}}^{*}([\mathrm{X}/\mu_\ell], \mathbb{Z}/\ell)^{\mathrm{Aut}(\mu_\ell)} \end{aligned}$$

are split injective. One may also observe that the action of $Aut(\mu_\ell)$ is compatible with the cycle map. Therefore the case of $[\mathrm{X}/\mu_\ell]$ considered above completes the proof of (iii) in the Theorem. □

6. Brauer group of the moduli stack of elliptic curves and Proof of Theorem 1.7.

Let $\mathcal{M}_{1,1,\mathrm{R}}$ denote the moduli stack of elliptic curves over the base scheme B, which we assume is the spectrum of a Dedekind domain R, for example, the ring of integers in a number field. We proceed to compute the ℓ-primary torsion part of the corresponding Brauer group. We will assume that the primes 2 and 3 are invertible in R. Then we make use of the following presentation for the stack $\mathcal{M}_{1,1,\mathrm{R}}$: see [Ols, Proposition 28.6] or [Hart77, Chapter IV section 4]. Let $\mathrm{Y} = \mathrm{Spec}\,\mathrm{R}[\mathrm{g}_2, \mathrm{g}_3][1/\Delta] \subseteq \mathbb{A}^2_{\mathrm{R}}$, where $\Delta = g_2^3 - 27g_3^2$. We define an action of $\mathbb{G}_m$ by $g_2 \mapsto u^4 g_2, g_3 \mapsto u^6 g_3$, $u \in \mathbb{G}_m$. Let ℓ denote any prime and let $\mathrm{B} = \mathrm{Spec}\,\mathrm{R}$.

$$(6.1)\qquad \mathcal{M}_{1,1,\mathrm{R}} = [\mathrm{Y}/\mathbb{G}_\mathrm{m}]$$

Then we obtain the following result..

Theorem 6.1. *Assume further that the prime ℓ is invertible in* R*. Then the following hold:*

(i) $\mathrm{Br}(\mathrm{Y})_{\ell^n} \cong \mathrm{Br}(\mathrm{B})_{\ell^n} \oplus \mathrm{H}^1_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^n})$.

(ii) $\mathrm{Br}(\mathcal{M}_{1,1,\mathrm{R}})_{\ell^n} = \mathrm{kernel}(\mathrm{res} : \mathrm{Br}((\mathbb{A}^1 \times \mathbb{G}_\mathrm{m}) \times_{\mathbb{G}_\mathrm{m}} \mathrm{Y})_{\ell^n} \cong \mathrm{Br}(\mathrm{Y})_{\ell^n} \to \mathrm{H}^3_{\mathrm{et},(\mathbb{G}_\mathrm{m} \times \{0\}) \times_{\mathbb{G}_\mathrm{m}} \mathrm{Y}}(\mathbb{G}_\mathrm{m} \times \mathbb{A}^1 \times_{\mathbb{G}_\mathrm{m}} \mathrm{Y}, \mu_{\ell^n}(1)) \cong \mathrm{H}^1_{\mathrm{et}}((\mathbb{G}_\mathrm{m} \times \{0\}) \times_{\mathbb{G}_\mathrm{m}} \mathrm{Y}, \mu_{\ell^n}) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{Y}, \mu_{\ell^n}))$*, where res denotes the residue map discussed in* (2.10) *and in* (2.14)*. In particular the Brauer group* $\mathrm{Br}(\mathcal{M}_{1,1,\mathrm{R}})_{\ell^n}$ *is a subgroup of* $\mathrm{Br}(\mathrm{Y})_{\ell^n}$*. Moreover* $\mathrm{Br}(\mathcal{M}_{1,1,\mathrm{R}})_{\ell^n} \cong 0$ *if the residue map in (ii) is injective.*

(iii) In case the Brauer group $\mathrm{Br}(\mathrm{B})_{\ell^n}$ *is trivial,* $\mathrm{Br}(\mathrm{Y})_{\ell^n} \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^n})$ *and hence* $\mathrm{Br}(\mathcal{M}_{1,1,\mathrm{R}})_{\ell^n}$ *is generated by classes coming from* $\mathrm{H}^1_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^n})$*.*

(iv) In case R *is a finite field or a complete discrete valuation ring with finite residue field,* $\mathrm{Br}(\mathrm{B})_{\ell^n}$ *is trivial, so that the conclusions in (iii) hold in these cases.*

Proof. Throughout the proof, we will let $x = g_2$, $y = g_3$ and $\tilde{\Delta} = Spec\,\mathrm{R}[x,y]/(x^3 - 27y^2)$. Let $\mathbb{A}^i_{\mathrm{B}}$ denote the affine space of dimension i over $\mathrm{B} = \mathrm{Spec}\,\mathrm{R}$. Observe that the curve corresponding to $\tilde{\Delta}$ has an isolated singularity at the origin, which can be resolved by taking the normalization as follows. Recall that $\tilde{\Delta}$ corresponds to the plane curve with equation : $(x/3)^3 = y^2$. Therefore, we substitute $(x/3) = t^2$ and $y = t^3$, so that $A = \mathrm{R}[x,y]/((x^3 - 27y^2) \cong \mathrm{R}[\mathrm{t}^2, \mathrm{t}^3]$ with function field K(t), where K denotes the function field of R. This is because $1/t = t^2/t^3 = (x/3)/y = x/(3y)$. Since R is assumed to be a Dedekind domain, it is integrally closed in its field of fractions K. Therefore, R[t] is integrally closed in K(t): see [StacksP, Normal rings: Lemma 10.7.8]. Clearly t is integral over A, and therefore the integral closure of A in K(t) is $\mathrm{R[t]} = \mathrm{R}[3y/x]$, which corresponds to the affine line $\mathbb{A}^1_{\mathrm{B}}$ over $\mathrm{B} = \mathrm{Spec}\,\mathrm{R}$. This proves that the normalization of the curve $\tilde{\Delta}$ is the affine line $\mathbb{A}^1_{\mathrm{B}}$ and the normalization maps $\mathbb{A}^1_{\mathrm{B}} - \{0\}$ isomorphically to the curve $\tilde{\Delta} - \{0\}$.

Therefore, we obtain the isomorphisms:

$$\mathbb{A}^2_{\mathrm{B}} - \tilde{\Delta} \cong (\mathbb{A}^2_{\mathrm{B}} - \{0\}) - ((\tilde{\Delta}) - \{0\}) \cong \mathbb{A}^2_{\mathrm{B}} - \mathbb{A}^1_{\mathrm{B}} \cong \mathbb{A}^1_{\mathrm{B}} \times \mathbb{G}_{m,\mathrm{B}} \tag{6.2}$$

$$\mathrm{H}^3_{\mathrm{et}}(\mathbb{A}^2_{\mathrm{B}} - \tilde{\Delta}, \mu_{\ell^{\mathrm{n}}}(1)) \cong \mathrm{H}^3_{\mathrm{et}}((\mathbb{A}^2_{\mathrm{B}} - \{0\}) - (\tilde{\Delta} - \{0\}), \mu_{\ell^{\mathrm{n}}}(1)) \cong \mathrm{H}^3_{\mathrm{et}}((\mathbb{A}^2_{\mathrm{B}} - \mathbb{A}^1_{\mathrm{B}}), \mu_{\ell^{\mathrm{n}}}(1)),$$

where the last isomorphism follows from the observation made earlier that the normalization of $\tilde{\Delta}$ is the affine line $\mathbb{A}^1$. At this point Corollary 2.3(iii) applies to provide a proof of the first assertion. The second assertion then follows from (2.10). The remaining statements are clear. □

Next one considers the following conditions on B.

(i) $\mathrm{Pic(B)} = 0$,

(ii) $\mathrm{H}^2_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}}}) = 0$. (In case ℓ is invertible in B, $\mathrm{H}^2_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}}}) = 0$.) and

(iii) $\mathrm{H}^1_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}}}) = 0$. (In case ℓ is invertible in B, $\mathrm{H}^1_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}}}) = 0$.)

Lemma 6.2. *(i) Under the hypothesis in (i),* $\mathrm{Br(B)}_{\ell^{\mathrm{n}}} \cong \mathrm{H}^2_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}}}(1))$*, which denotes cohomology computed on the fppf site. Therefore, under the hypothesis in (i), (ii) is equivalent to* $\mathrm{Br(B)}_{\ell^{\mathrm{n}}} \cong 0$.

(ii) Under the hypothesis in (i), $\mathrm{H}^1_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}}}) \cong 0$ *if and only the the ℓ^n-th power map* $\Gamma(\mathrm{B}, \mathbb{G}_{\mathrm{m}}) \rightarrow \Gamma(\mathrm{S}, \mathbb{G}_{\mathrm{m}})$ *is surjective.*

Proof. To see (i), consider the the following part of the long exact sequence provided by the Kummer sequence:

$$\cdots \rightarrow \mathrm{H}^1_{\mathrm{fppf}}(\mathrm{B}, \mathbb{G}_{\mathrm{m}}) \rightarrow \mathrm{H}^1_{\mathrm{fppf}}(\mathrm{B}, \mathbb{G}_{\mathrm{m}}) \overset{\alpha}{\rightarrow} \mathrm{H}^2_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}}}) \overset{\beta}{\rightarrow} \mathrm{H}^2_{\mathrm{fppf}}(\mathrm{B}, \mathbb{G}_{\mathrm{m}}) \overset{\gamma}{\rightarrow} \mathrm{H}^2_{\mathrm{fppf}}(\mathrm{B}, \mathbb{G}_{\mathrm{m}}) \rightarrow \cdots \tag{6.3}$$

(If ℓ is invertible in B, one may also use the corresponding long-exact sequence in étale cohomology.) The map denoted γ is the ℓ^n-th power map, and its kernel is $\mathrm{Br(B)}_{\ell^{\mathrm{n}}}$. The exactness of the long exact sequence above shows that the kernel of γ is isomorphic to the image of β. But the map β is clearly injective in view of the assumption that $\mathrm{Pic(B)} = 0$. Thus $\mathrm{Br(B)}_{\ell^{\mathrm{n}}} \cong \mathrm{H}^2_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}}})$.

For (ii) we consider the following part of the long exact sequence provided by the Kummer sequence:

$$0 \rightarrow \mathrm{H}^0_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}}}) \rightarrow \Gamma(\mathrm{B}, \mathbb{G}_{\mathrm{m}}) \overset{\ell^{\mathrm{n}}}{\rightarrow} \Gamma(\mathrm{B}, \mathbb{G}_{\mathrm{m}}) \overset{\delta}{\rightarrow} \mathrm{H}^1_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}}}) \rightarrow \mathrm{Pic(B)} \cong 0 \tag{6.4}$$

This shows that $\mathrm{H}^1_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}}})$ is isomorphic to the cokernel of the ℓ^n-th power map. Thus $\mathrm{H}^1_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}}}) \cong 0$ if and only if the ℓ^n-th power map $\Gamma(\mathrm{B}, \mathbb{G}_{\mathrm{m}}) \overset{\ell^{\mathrm{n}}}{\rightarrow} \Gamma(\mathrm{B}, \mathbb{G}_{\mathrm{m}})$ is surjective. □

Next we make the following observations. Let $\mathbb{Z}$ denote the ring of integers in $\mathbb{Q}$. Then

(i) $\mathrm{Pic}(\mathrm{Spec}\,\mathbb{Z}) \cong 0$, and

(ii) $\mathrm{Br}(\mathrm{Spec}\,\mathbb{Z}) \cong 0$. In particular $\mathrm{Br}(\mathrm{Spec}\,\mathbb{Z})_{\ell^{\mathrm{n}}} \cong 0$.

One may readily see (i) is true because $\mathbb{Z}$ is a PID. The statement that $\mathrm{Br}(\mathrm{Spec}\,\mathbb{Z}) \cong 0$ may be proven using class-field-theory: see [Ay], for example. Since $\mathrm{Br}(\mathrm{Spec}\,\mathbb{Z})_{\ell^{\mathrm{n}}}$ denotes the ℓ^n-torsion part of $\mathrm{Br}(\mathrm{Spec}\,\mathbb{Z})$ the second assertion in (ii) follows.

Next observe that $\mathrm{Spec}\,\mathbb{Z}[1/6] = \mathrm{Spec}\,\mathbb{Z} - \{(2), (3)\}$. Therefore, one may compute its Brauer group as follows.

Lemma 6.3. $\mathrm{Br}(\mathrm{Spec}\,\mathbb{Z}[1/6])_{\ell^n} \cong (\mathbb{Z}/2\mathbb{Z})_{\ell^n} \oplus (\mathbb{Q}/\mathbb{Z})_{\ell^n}$, *for* $\ell = 2$ *or* 3.

Proof. We skip the proof as this follows from a standard argument using the computation that $\mathrm{Br}(\mathbb{Z}) = 0$. □

Taking $\mathrm{R} = \mathbb{Z}[1/6]$, we obtain the following corollary to Theorem 6.1.

Corollary 6.4. *Let* $\mathrm{B} = \mathrm{Spec}\,\mathbb{Z}[1/6]$ *and let* Y *be as in Example 6. Let* $\mathbb{Z}[1/6]^*$ *denote the units in the ring* $\mathbb{Z}[1/6]$. *Then for* $\ell = 2$, *or* 3,

(i) $\mathrm{Br}(\mathrm{Y})_{\ell^n} \cong (\mathbb{Z}/2\mathbb{Z})_{\ell^n} \oplus (\mathbb{Q}/\mathbb{Z})_{\ell^n} \oplus (\mathrm{coker}(\mathbb{Z}[1/6]^* \xrightarrow{\ell^n} \mathbb{Z}[1/6]^*)$,

(ii) $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Y}, \mu_{\ell^n}) \cong (\mathrm{coker}(\mathbb{Z}[1/6]^* \xrightarrow{\ell^n} \mathbb{Z}[1/6]^*) \oplus (\ker(\mathbb{Z}[1/6]^* \xrightarrow{\ell^n} \mathbb{Z}[1/6]^*)$, *and*

(iii) $\mathrm{Br}(\mathcal{M}_{1,1,\mathrm{R}})_{\ell^n} = \ker(\mathrm{res} : \mathrm{Br}(\mathrm{Y})_{\ell^n} \to \mathrm{H}^1_{\mathrm{et}}(\mathrm{Y}, \mu_{\ell^n}))$, *where res denotes the residue map as in Theorem 6.1.*

Proof. The Kummer sequence in (6.4) (which holds on the étale site in this case since $\ell = 2, 3$) shows that $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,\mathbb{Z}[1/6], \mu_{\ell^n}) \cong \mathrm{coker}(\mathbb{Z}[1/6]^* \xrightarrow{\ell^n} \mathbb{Z}[1/6]^*)$. Therefore (i) follows. The Kummer sequence in (6.4) shows that $\mathrm{H}^0_{\mathrm{et}}(\mathrm{Spec}\,\mathbb{Z}[1/6], \mu_\ell) \cong \ker(\mathbb{Z}[1/6]^* \xrightarrow{\ell^n} \mathbb{Z}[1/6]^*)$ Therefore (ii) follows. Now (iii) follows readily from Theorem 6.1 and the above observations. This completes the proof of the Corollary. □

7. Brauer group and torsion index of linear algebraic groups: Proof of Theorem 1.8.

We next recall the definition of the *torsion index* of connected linear algebraic groups from [Tot05, section 1]. Throughout this section we will worked over any base field k with ℓ a prime different from $char(k)$, and that all linear algebraic groups we consider are defined over k. Let H denote a fixed connected linear algebraic group with a chosen Borel subgroup B and a chosen maximal torus $\mathrm{T} \subseteq \mathrm{B}$. Let N denote the dimension of H/B. For a linear algebraic group G, we will let $\mathrm{BG}^{\mathrm{gm}}$ denote $\mathrm{BG}^{\mathrm{gm},m}$, for some $m >> 0$.

Next consider the diagram $\mathrm{H/B} \xrightarrow{\mathrm{i}} \mathrm{BB}^{\mathrm{gm}} \xrightarrow{\mathrm{f}} \mathrm{BH}^{\mathrm{gm}}$, where f denotes the obvious map induced by the inclusion $\mathrm{B} \subseteq \mathrm{H}$. Observe that $\mathrm{BB}^{\mathrm{gm}} \simeq \mathrm{BT}^{\mathrm{gm}}$, where $\simeq$ denotes a weak-equivalence in the motivic homotopy category.

Definition 7.1. *(See* [Tot05, section 1] *and also* [Tot05.1, Definition 1]*.) Let* $\mathrm{N} = \dim(\mathrm{H/B})$*, so that* $\mathrm{CH}^{\mathrm{N}}(\mathrm{H/B}) \cong \mathbb{Z}$. *Then the torsion index of* H *(denoted* $t(\mathrm{H})$*) is the positive integer* $t(\mathrm{H})$ *so that*

$$image(i^* : \mathrm{CH}^{\mathrm{N}}(\mathrm{BB}^{\mathrm{gm}}) \to \mathrm{CH}^{\mathrm{N}}(\mathrm{H/B}))$$

is $t(\mathrm{H}).\mathrm{CH}^{\mathrm{N}}(\mathrm{H/B})$.

Observe that there exists a class $a \in \mathrm{CH}^{\mathrm{N}}(\mathrm{BB}^{\mathrm{gm}}, \mathrm{Z}/\ell^{\mathrm{n}}) (\cong \mathrm{H}^{2\mathrm{N},\mathrm{N}}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm}}, \mathbb{Z}/\ell^{\mathrm{n}}))$ so that

$$f_*(a) = \overline{t(\mathrm{H})} \in \mathrm{CH}^0(\mathrm{BH}^{\mathrm{gm}}, \mathrm{Z}/\ell^{\mathrm{n}}) \cong \mathrm{Z}/\ell^{\mathrm{n}}, \tag{7.1}$$

where $\overline{t(\mathrm{H})}$ is the image of the torsion index $t(\mathrm{H})$.

Remark 7.2. *For linear algebraic groups defined over the complex numbers, the torsion index can also be defined using singular cohomology with integral coefficients, as for such groups the singular cohomology of* H/B *with integral coefficients is isomorphic to the corresponding Chow group.*

Next we consider the following squares that commute:

$$\begin{array}{ccc} \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm}}, \mathbb{Z}/\ell^{\mathrm{n}}) & \xrightarrow{f_*} & \mathrm{H}^{*-2\mathrm{N},\bullet-\mathrm{N}}_{\mathrm{M}}(\mathrm{BH}^{\mathrm{gm}}, \mathbb{Z}/\ell^{\mathrm{n}}) \\ \downarrow{\scriptstyle cycl} & & \downarrow{\scriptstyle cycl} \\ \mathrm{H}^*_{\mathrm{et}}(\mathrm{BB}^{\mathrm{gm}}, \mu_{\ell^n}(\bullet)) & \xrightarrow{\bar{f}_*} & \mathrm{H}^{*-2\mathrm{N}}_{\mathrm{et}}(\mathrm{BH}^{\mathrm{gm}}, \mu_{\ell^n}(\bullet - \mathrm{N})) \end{array} \tag{7.2}$$

$$\begin{array}{ccc} \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm}}, \mathbb{Z}/\ell^{\mathrm{n}}) & \xleftarrow[f^*]{} & \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{BH}^{\mathrm{gm}}, \mathbb{Z}/\ell^{\mathrm{n}}) \\ \downarrow{\scriptstyle cycl} & & \downarrow{\scriptstyle cycl} \\ \mathrm{H}^{*}_{\mathrm{et}}(\mathrm{BB}^{\mathrm{gm}}, \mu_{\ell^{\mathrm{n}}}(\bullet)) & \xleftarrow[\bar{f}^*]{} & \mathrm{H}^{*}_{\mathrm{et}}(\mathrm{BH}^{\mathrm{gm}}, \mu_{\ell^{\mathrm{n}}}(\bullet)) \end{array} \tag{7.3}$$

To see that these squares commute, one may observe that $\mathrm{BB}^{\mathrm{gm,m}}$ identifies with $\mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} (\mathrm{H/B})$ while $\mathrm{BH}^{\mathrm{gm,m}}$ identifies with $\mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} (\mathrm{H/H}) \cong \mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} (\mathrm{Spec}\, k)$. Now observe that, since B is a Borel subgroup in H, H/B is a flag variety and therefore admits an H-equivariant closed immersion into a projective space $\mathbb{P}^n$ on which H acts. Therefore, the map $\mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} \mathrm{H/B} \to \mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} \mathrm{H/H} \cong \mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} \mathrm{Spec}\, k$ factors as the composition $\mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} \mathrm{H/B} \xrightarrow{i} \mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} \mathbb{P}^{\mathrm{n}} \xrightarrow{\pi} \mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} \mathrm{Spec}\, k$. Therefore it suffices to show that the cycle map commutes with i_* and i^* as well as with π_* and π^*. One may prove i_* commutes with the cycle map by using a deformation to the normal cone argument. To prove π_* commutes withe the cycle map, one may use the projective space bundle formula for motivic and étale cohomology. The commutativity of i^* and π^* with the cycle map may be proven similarly.

Lemma 7.3. *(i) The cycle map* $cycl : \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm}}, \mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}^{*}_{\mathrm{et}}(\mathrm{BB}^{\mathrm{gm}}, \mu_{\ell^{\mathrm{n}}}(\bullet))$ *is an isomorphism when the base field k is separably closed.*

(ii) For any base field k, $\mathrm{H}^{u}_{\mathrm{et}}(\mathrm{BB}^{\mathrm{gm}}, \mu_{\ell^{\mathrm{n}}}(v)) \cong \oplus_{i+2m=u, j+m=v} \mathrm{H}^{i}_{\mathrm{et}}(\mathrm{Spec}\, k, \mu_{\ell^{\mathrm{n}}}(j)) \otimes_{\mathbb{Z}/\ell^n} \mathrm{H}^{2m}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm}}, \mathbb{Z}/\ell^{\mathrm{n}}(m))$. *In particular,*

$$\begin{aligned} \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{BB}^{\mathrm{gm}}, \mu_{\ell^{\mathrm{n}}}(1)) &\cong \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{Spec}\, k, \mu_{\ell^{\mathrm{n}}}(1)) \otimes_{\mathbb{Z}/\ell^n} \mathrm{H}^{0}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm}}, \mathbb{Z}/\ell^{\mathrm{n}}(0)) \oplus \mathrm{H}^{0}_{\mathrm{et}}(\mathrm{Spec}\, k, \mu_{\ell^{\mathrm{n}}}(0)) \otimes_{\mathbb{Z}/\ell^n} \mathrm{H}^{2}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm}}, \mathbb{Z}/\ell^{\mathrm{n}}(1)) \\ &\cong \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{Spec}\, k, \mu_{\ell^{\mathrm{n}}}(1)) \oplus \mathrm{H}^{2}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm}}, \mathbb{Z}/\ell^{\mathrm{n}}(1)). \end{aligned}$$

(iii) For any base field k, the cycle map in the left column in the commutative squares (7.2) *and* (7.3) *are both injective.*

(iv) $\mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{Spec}\, k, \mathbb{Z}/\ell^{\mathrm{n}})$ *is a split summand of both* $\mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm}}, \mathbb{Z}/\ell^{\mathrm{n}})$ *and* $\mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{BH}^{\mathrm{gm}}, \mathbb{Z}/\ell^{\mathrm{n}})$. $\mathrm{H}^{*}_{\mathrm{et}}(\mathrm{Spec}\, k, \mu_{\ell^{\mathrm{n}}}(\bullet))$ *is a split summand of both* $\mathrm{H}^{*}_{\mathrm{et}}(\mathrm{BB}^{\mathrm{gm}}, \mu_{\ell^{\mathrm{n}}}(\bullet))$ *and* $\mathrm{H}^{*}_{\mathrm{et}}(\mathrm{BH}^{\mathrm{gm}}, \mu_{\ell^{\mathrm{n}}}(\bullet))$. *It follows that* $\mathrm{Br}(\mathrm{Spec}\, k)_{\ell^{\mathrm{n}}}$ *is a split summand of both* $\mathrm{Br}(\mathrm{BB}^{\mathrm{gm}})_{\ell^{\mathrm{n}}}$ *and* $\mathrm{Br}(\mathrm{BH}^{\mathrm{gm}})_{\ell^{\mathrm{n}}}$.

Proof. Observe that $\mathrm{BB}^{\mathrm{gm}} \simeq \mathrm{BT}^{\mathrm{gm}} = \mathbb{P}^{\infty}$, where T is a split maximal torus in H. Therefore, the first two statements follow readily from the calculation of the motivic and étale cohomology of a projective space $\mathbb{P}^n$. In fact, we will presently provide the following details on this. One starts with the isomorphisms provided by the projective space bundle formula:

$$\mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm,m}}, \mathbb{Z}/\ell^{\mathrm{n}}) \cong \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{Spec}\, k, \mathbb{Z}/\ell^{\mathrm{n}})[\mathrm{t}]/(\mathrm{t}^{\mathrm{m}+1}) \cong \mathrm{H}^{*,\bullet}(\mathrm{Spec}\, k, \mathbb{Z}/\ell^{\mathrm{n}}) \underset{\mathbb{Z}/\ell^{\mathrm{n}}}{\otimes} \mathbb{Z}/\ell^{\mathrm{n}}[\mathrm{t}]/(\mathrm{t}^{\mathrm{m}+1}), \tag{7.4}$$

where t has bi-degree $(2, 1)$. Similarly one see that

$$\mathrm{H}^{*}_{\mathrm{et}}(\mathrm{BB}^{\mathrm{gm,m}}, \mu_{\ell^{\mathrm{n}}}(\bullet)) \cong \mathrm{H}^{*}_{\mathrm{et}}(\mathrm{Spec}\, k, \mu_{\ell^{\mathrm{n}}}(\bullet))[\mathrm{t}]/(\mathrm{t}^{\mathrm{m}+1}) \cong \mathrm{H}^{*,\bullet}_{\mathrm{et}}(\mathrm{Spec}\, k, \mu_{\ell^{\mathrm{n}}}(\bullet)) \underset{\mathbb{Z}/\ell^{\mathrm{n}}}{\otimes} \mathbb{Z}/\ell^{\mathrm{n}}[\mathrm{t}]/(\mathrm{t}^{\mathrm{m}+1}), \tag{7.5}$$

where t has bi-degree $(2, 1)$. Next one may observe that $\oplus_{0 \le j \le m} \mathrm{H}^{2\mathrm{j,j}}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm,m}}, \mathbb{Z}/\ell^{\mathrm{n}}) \cong \mathbb{Z}/\ell^{\mathrm{n}}[\mathrm{t}]/(\mathrm{t}^{\mathrm{m}+1})$. These observations prove the first statement in (ii). The second statement in (ii) is now an immediate consequence of the first statement.

The third statement is an immediate consequence of the first two. Observe from the definition of admissible gadgets as in (3.1), we require that the U_{m} there always has a k-rational point. It follows that the finite degree approximations $\mathrm{BB^{gm,m}}$ and $\mathrm{BH^{gm,m}}$ also have k-rational points. The statements in (iv) are immediate consequences of this. □

Moreover, one may observe that the cycle map $\mathrm{H_M^{0,0}(BH^{gm}, \mathbb{Z}/\ell^n) \to H^0_{et}(BH^{gm}, \mu_{\ell^n}(0)) \cong Z/\ell^n}$ is also an isomorphism. In view of these observations, one may define the torsion index $\overline{t(\mathrm{H})}$ as the class in $\mathbb{Z}/\ell^n$ so that if $\bar{a} = cycl(a)$, with the class a as in (7.1)

$$\bar{f}_*(\bar{a}) = \overline{t(\mathrm{H})} \in \mathrm{H}^0_{et}(\mathrm{BH^{gm}}, \mu_{\ell^n}) \cong \mathrm{Z}/\ell^{\mathrm{n}}, \tag{7.6}$$

Proposition 7.4. *(See* [Tot05, section 1]*.) (i) The kernel of the cycle map*

$$cycl : \mathrm{H_M^{*,\bullet}(BH^{gm}, \mathbb{Z}/\ell^n) \to H^*_{et}(BH^{gm}, \mu_{\ell^n})},$$

as well as the kernel of the restriction map

$$f^* : \mathrm{H_M^{*,\bullet}(BH^{gm}, \mathbb{Z}/\ell^n) \to H_M^{*,\bullet}(BB^{gm}, \mathbb{Z}/\ell^n)}$$

are killed by $t(\mathrm{H})$*.*

(ii) $\mathrm{Br(BH^{gm})_{\ell^n}/(Br(Spec\, k)_{\ell^n}) = H^2_{et}(BH^{gm}.\mu_{\ell^n}(1))/(Im(cycl) + H^2_{et}(Spec\, k, \mu_{\ell^n}(1)))}$ *is killed by* $t(\mathrm{H})$*, where* $\mathrm{Im}(cycl)$ *denotes the image of the cycle map* $cycl : \mathrm{H_M^{2,1}(BH^{gm}, \mathbb{Z}/\ell^n) \to H^2_{et}(BH^{gm}, \mu_{\ell^n}(1))}$*.*

Proof. Define a map $\alpha : \mathrm{CH^i(BB^{gm}, \mathbb{Z}/\ell^n) \cong H_M^{2i,i}(BB^{gm}, \mathbb{Z}/\ell^n) \to CH^i(BH^{gm}, \mathbb{Z}/\ell^n) \cong H_M^{2i,i}(BH^{gm}, \mathbb{Z}/\ell^n)}$ by $\alpha(x) = f_*(a.x)$. Then, $\alpha(f^*(x)) = f_*(a.f^*(x)) = f_*(a).x = t(\mathrm{H}).x$. As $\mathrm{BT^{gm}}$ identifies with $\mathrm{BB^{gm}}$, the map f^* identifies with the restriction homomorphism $res : \mathrm{H_M^{*,\bullet}(BH^{gm}, \mathbb{Z}/\ell^n) \to H_M^{*,\bullet}(BT^{gm}, \mathbb{Z}/\ell^n)}$, thereby proving that its kernel is killed by the class $t(\mathrm{H})$. In view of the fact that cycle map forming the left vertical map in (7.3) is injective, it follows that the kernel of the cycle map

$$cycl : \mathrm{H_M^{*,\bullet}(BH^{gm}, \mathbb{Z}/\ell^n) \to H^*_{et}(BH^{gm}, \mu_{\ell^n}(\bullet))} \tag{7.7}$$

is contained in the kernel of f^*, and hence is killed by the class $t(\mathrm{H})$. This completes the proof of (i).

We next consider the statement in (ii). Therefore, let $\bar{x} \in \mathrm{H^2_{et}(BH^{gm}, \mu_{\ell^n}(1)}$ denote a class. Then

$$t(\mathrm{H}).\bar{x} = \mathrm{\bar{f}_*(cycl(a).\bar{f}^*(\bar{x}))}. \tag{7.8}$$

In view of statements (ii) and (iv) in Lemma 7.3, one may write

$$\bar{f}^*(\bar{x}) = cycl(y) + z = cycl(y) + \bar{f}^*(\bar{z}), y \in \mathrm{H_M^{2,1}(BB^{gm}, \mathbb{Z}/\ell^n)}, \bar{z} \in \mathrm{H^2_{et}(Spec\, k, \mu_{\ell^n}(1))} \tag{7.9}$$

Therefore,

$$t(\mathrm{H}).\bar{x} = \bar{f}_*(\mathrm{cycl(a}.y) + \mathrm{cycl(a)}.\bar{f}^*(\bar{z})) = \mathrm{cycl}(\bar{f}_*(\mathrm{a}.y)) + \bar{f}_*(\mathrm{cycl(a)}.\bar{f}^*\bar{z}) = \mathrm{cycl}(\bar{f}_*(\mathrm{a}.y)) + \mathrm{t(H)}\bar{z}. \tag{7.10}$$

This proves the statement in (ii), thereby completing the proof of the Proposition. □

7.1. **Information on the torsion index.** Using the fact that connected reductive groups over any algebraically closed field of positive characteristic admit liftings to characteristic 0, one may make use of the determination of the torsion index for compact Lie groups. The following are known:

(i) The only primes that divide the torsion index of simply-connected groups are $2, 3$ and 5.
(ii) The torsion index for GL_n, SL_n and $\mathrm{Sp}(2n)$, for any n is 1.
(iii) The torsion index for $\mathrm{SO}(2n)$ and $\mathrm{SO}(2n+1)$, for any n, are powers of 2.
(iv) The torsion index of $\mathrm{Spin}(n)$ is a power of 2: see [Tot05] for more details.

(v) The torsion index of E6 is 6 and of E8 is $2^6 3^2 5$: see [Tot05.2].

Corollary 7.5. *(i)* $\mathrm{Br(BG)}_{\ell^n} \cong \mathrm{Br}(\mathrm{Spec}\, k)_{\ell^n}$ *for any prime* ℓ *different from the characteristic of* k *if* $\mathrm{G} = \mathrm{GL_n}$, $\mathrm{G} = \mathrm{SL_n}$ *or* $\mathrm{G} = \mathrm{Sp(2n)}$, *for any* n.

(ii) $\mathrm{Br(BG)}_{\ell^n} \cong \mathrm{Br}(\mathrm{Spec}\, k)_{\ell^n}$ *for any prime* ℓ *different from the characteristic of* k *and* 2 *if* $\mathrm{G} = \mathrm{SO(2n)}$, $\mathrm{SO}(2n+1)$, *or* $\mathrm{Spin}(n)$, *for any* n.

(iii) $\mathrm{Br(BG)}_{\ell^n} \cong \mathrm{Br}(\mathrm{Spec}\, k)_{\ell^n}$ *for any prime* ℓ *different from the characteristic of* k *for any simply-connected group* G, *if* ℓ *is also different from* $2, 3,$ *or* 5.

Proof of Theorem 1.8. Clearly the above discussion completes the proof of the theorem. □

8. Proof of Theorem 1.9.

We first observe that $\mathrm{B}\mathbb{G}_m^{\mathrm{gm}} \cong \lim_{n\to\infty} \mathbb{P}^n$. Since each $\mathbb{P}^n$ is a linear scheme which is projective and smooth, it follows from [J01, Theorem 4.5, Corollary 4.6] that one obtains isomorphisms for any smooth scheme Y:

$$\begin{aligned} &\oplus_i \mathrm{H}_{\mathrm{M}}^{2\mathrm{i,i}}(\mathrm{B}\mathbb{G}_{\mathrm{m}}^{\mathrm{gm}} \times \mathrm{Y}, \mathbb{Z}/\ell^{\mathrm{n}}) \cong (\oplus_i \mathrm{H}_{\mathrm{M}}^{2\mathrm{i,i}}(\mathrm{B}\mathbb{G}_{\mathrm{m}}, \mathbb{Z}/\ell^{\mathrm{n}})) \otimes (\oplus_{\mathrm{i}} \mathrm{H}_{\mathrm{M}}^{2\mathrm{i,i}}(\mathrm{Y}, \mathbb{Z}/\ell^{\mathrm{n}})) \text{ and} \\ &\oplus_i \mathrm{H}_{\mathrm{et}}^{2\mathrm{i}}(\mathrm{B}\mathbb{G}_{\mathrm{m}}^{\mathrm{gm}} \times \mathrm{Y}, \mu_{\ell^{\mathrm{n}}}(\mathrm{i})) \cong (\oplus_i \mathrm{H}_{\mathrm{et}}^{2\mathrm{i}}(\mathrm{B}\mathbb{G}_{\mathrm{m}}, \mu_{\ell^{\mathrm{n}}}(\mathrm{i}))) \otimes (\oplus_{\mathrm{i}} \mathrm{H}_{\mathrm{et}}^{2\mathrm{i}}(\mathrm{Y}, \mu_{\ell^{\mathrm{n}}}(\mathrm{i}))). \end{aligned} \tag{8.1}$$

Since the cycle map $cycl : \oplus_i \mathrm{H}_{\mathrm{M}}^{2\mathrm{i,i}}(\mathrm{B}\mathbb{G}_{\mathrm{m}}^{\mathrm{gm}}, \mathbb{Z}/\ell^{\mathrm{n}}) \to \oplus_{\mathrm{i}} \mathrm{H}_{\mathrm{et}}^{2\mathrm{i}}(\mathrm{B}\mathbb{G}_{\mathrm{m}}^{\mathrm{gm}}, \mu_{\ell^{\mathrm{n}}}(\mathrm{i}))$ is an isomorphism, the Brauer group $\mathrm{Br}(\mathrm{Bun}_{1,\mathrm{d}}(\mathrm{X}))_{\ell^{\mathrm{n}}}$, which is the cokernel of cycle map, identifies with $\mathrm{Br}(\mathbf{P}\mathrm{ic}^{\mathrm{d}}(\mathrm{X}))_{\ell^{\mathrm{n}}}$. Finally the isomorphism $\mathrm{Br}(\mathbf{P}\mathrm{ic}^{\mathrm{d}}(\mathrm{X}))_{\ell^{\mathrm{n}}} \cong \mathrm{Br}(\mathrm{Sym}^{\mathrm{d}}(\mathrm{X}))_{\ell^{\mathrm{n}}}$ is proven in [IJ20, Theorem 1.2].

Recall that $\mathrm{Br(Y)} = 0$ if Y is a connected projective smooth variety that is *rational*: this follows from the well-known fact that the Brauer group is a stable birational invariant for connected projective smooth varieties. The last statement follows from this observation. □

9. Appendix: Motivic cohomology over regular Noetherian base schemes

First one may observe that the higher cycle complex may be defined over any base scheme B: if X is a scheme of finite type over B, one considers $Z^c(X,.)$ to be the chain complex defined in degree n, by

$$\{\mathrm{Z} = \text{ a pure codim c cycle on } \mathrm{X} \times_{\mathrm{B}} \Delta_{\mathrm{B}}[\mathrm{n}] | \mathrm{Z} \text{ intersects the faces of } \mathrm{X} \times \Delta_{\mathrm{B}}[\mathrm{n}] \text{ properly}\}. \tag{9.1}$$

Definition 9.1. *We let $\mathbb{Z}(c)$ denote the co-chain complex $Z^c(X,.)[-2c]$ in cohomological degree m, that is, $\mathbb{Z}(c) = Z^c(X, 2c-m)$. (Observe that $\mathcal{Z}(c)$ is contravariantly functorial for flat maps.) If* X *denotes a smooth scheme of finite type over* B*, we will let $\mathbb{Z}^{\mathrm{X}}(c)$ denote the restriction of the complex $\mathbb{Z}(c)$ to the small Zariski or Nisnevich site of* X*. We call the complex $\mathbb{Z}(c)$ the motivic complex of weight c.*

Next we will assume that $\mathrm{B} = \mathrm{Spec}\,\mathrm{R}$, where R is a Dedekind domain.

Proposition 9.2. *Assume in addition that* X *denotes a smooth scheme of finite type over* B*. Then $\mathbb{Z}^{\mathrm{X}}(1)[1]$ identifies with $\mathbb{G}_m^{\mathrm{X}}$, which denotes the restriction of the sheaf $\mathbb{G}_m$ to the small Nisnevich site of* X*.*

Proof. This is discussed in [Bl, section 6], where the discussion does not assume the base scheme is a field. One may also see [Spitz, section 7.3], where these results are reworked in the setting of motivic spectra. □

Definition 9.3. *(Motivic cohomology) Let* X *denote a scheme of finite type over* B*. We let $\mathrm{H}^{\mathrm{i,j}}_{\mathrm{M}}(\mathrm{X}) = \mathrm{H}^{\mathrm{i}}_{\mathrm{Zar}}(\mathrm{X}, \mathbb{Z}(\mathrm{j}))$, where $\mathrm{H}_{\mathrm{Zar}}$ denotes the hypercohomology on the Zariski site.*

It is observed in [Geis, Corollary 3.3], that one obtains the identification $\mathrm{H}^{\mathrm{i}}_{\mathrm{Zar}}(\mathrm{X}, \mathbb{Z}(\mathrm{j})) \cong \mathrm{H}_{\mathrm{Zar}}(\mathrm{B}, \pi_*(\mathbb{Z}(\mathrm{j})))$, where $\pi : \mathrm{X} \to \mathrm{B}$ denotes the structure map.

Let X denote a scheme of finite type over B and let Z denote a closed subscheme of X of pure codimension c with ope complement U. Let $i : \mathrm{Z} \to \mathrm{X}$ and $j : \mathrm{U} \to \mathrm{X}$ denote the corresponding immersions. Then it is shown in [Geis, Corollary 3.3] that one obtains the distinguished triangle

$$0 \to i_*\mathbb{Z}^{\mathrm{Z}}(n-c)[-2c] \to \mathbb{Z}^{\mathrm{X}}(n) \to j_*\mathbb{Z}^{\mathrm{U}}(n) \tag{9.2}$$

in the derived category of Zariski sheaves on X. In particular, this provides the identification of the terms in the long-exact sequence forming the top row in the diagram:

$$\begin{array}{ccccc} \mathrm{H}^{\mathrm{i}}_{\mathrm{Z}}(\mathrm{X}, \mathbb{Z}(\mathrm{n})) & \longrightarrow & \mathrm{H}^{\mathrm{i}}(\mathrm{X}, \mathbb{Z}(\mathrm{n})) & \longrightarrow & \mathrm{H}^{\mathrm{i}}(\mathrm{U}, \mathbb{Z}(\mathrm{n})) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathrm{H}^{\mathrm{i}}_{\mathrm{Zar}}(\mathrm{X}, \mathrm{i}_*\mathbb{Z}^{\mathrm{Z}}(\mathrm{n}-\mathrm{c})) & \longrightarrow & \mathrm{H}^{\mathrm{i}}_{\mathrm{Zar}}(\mathrm{X}, \mathbb{Z}^{\mathrm{X}}(\mathrm{n})) & \longrightarrow & \mathrm{H}^{\mathrm{i}}_{\mathrm{Zar}}(\mathrm{X}, \mathrm{j}_*\mathbb{Z}^{\mathrm{X}}(\mathrm{n})) \end{array} \tag{9.3}$$

Proposition 9.4. *Assume the above situation. If $n < c = codim_{\mathrm{X}}(\mathrm{Z})$, where $codim_{\mathrm{X}}(\mathrm{Z})$ denotes the codimension of* Z *in* X*, then $\mathrm{H}^{\mathrm{i}}_{\mathrm{Z}}(\mathrm{X}, \mathbb{Z}(\mathrm{n})) = 0$.*

Proof. This is clear from the identification of the first terms in the commutative diagram (9.3), since when $n < c$, the complex $\mathbb{Z}^{\mathrm{Z}}(n-c)$ is trivial. □

9.1. **Some technical results.** We will conclude this section with the following technical results that will be used elsewhere in the paper in computing the Brauer groups. We begin with the spectral sequences:

$$\begin{aligned} &\mathrm{E}_2^{\mathrm{s,t}} = \mathrm{H}^{\mathrm{s}}_{\mathrm{Nis}}(\mathrm{U}, \mathrm{R}^{\mathrm{t}}\mathrm{p}_*(\tau_{\le \mathrm{j}}\mathrm{R}\epsilon_*\epsilon^*(\mathbb{Z}/\ell^{\mathrm{n}}(\mathrm{j})))) \Rightarrow \mathrm{H}^{\mathrm{s+t}}_{\mathrm{Nis}}(\mathrm{p}^{-1}(\mathrm{U}), \mathbb{Z}/\ell^{\mathrm{n}}(\mathrm{j})) \\ &\mathrm{E}_2^{\mathrm{s,t}} = \mathrm{H}^{\mathrm{s}}_{\mathrm{Nis}}(\mathrm{U}, \mathrm{R}^{\mathrm{t}}\mathrm{p}_*(\mathrm{R}\epsilon_*\epsilon^*(\mathbb{Z}/\ell^{\mathrm{n}}(\mathrm{j})))) \Rightarrow \mathrm{H}^{\mathrm{s+t}}_{\mathrm{Nis}}(\mathrm{p}^{-1}(\mathrm{U}), \mathrm{R}\epsilon_*\epsilon^*(\mathbb{Z}/\ell^{\mathrm{n}}(\mathrm{j}))) \cong \mathrm{H}^{\mathrm{s+t}}_{\mathrm{et}}(\mathrm{p}^{-1}(\mathrm{U}), \mu_{\ell^{\mathrm{n}}}(\mathrm{j})) \end{aligned} \tag{9.4}$$

The obvious map from the first spectral sequence to the second induces an isomorphism on the E_2-terms for $0 \leq s+t \leq j$, as $s \geq 0$.

Lemma 9.5. *Let $p : \mathcal{X} \to \mathcal{Y}$ denote a map of smooth schemes over k, so that it is Zariski locally trivial, with fibers given by the scheme* X *satisfying the condition that the cycle map:*

$$cycl : \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(1))$$

is an isomorphism. Let U, V *denote two Zariski open subschemes of $\mathcal{Y}$ so that $\mathcal{X} \times_{\mathcal{Y}} \mathrm{U} \cong \mathrm{U} \times \mathrm{X}$ and $\mathcal{X} \times_{\mathcal{Y}} \mathrm{V} \cong \mathrm{V} \times \mathrm{X}$. Assume that the corresponding cycle maps*

$$\mathrm{H}^{2,1}_{\mathrm{M}}(\mathcal{X} \times_{\mathcal{Y}} \mathrm{U}, \mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}^2_{\mathrm{et}}(\mathcal{X} \times_{\mathcal{Y}} \mathrm{U}, \mu_{\ell^{\mathrm{n}}}(1)) \text{ and } \mathrm{H}^{2,1}_{\mathrm{M}}(\mathcal{X} \times_{\mathcal{Y}} \mathrm{V}, \mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}^2_{\mathrm{et}}(\mathcal{X} \times_{\mathcal{Y}} \mathrm{V}, \mu_{\ell^{\mathrm{n}}}(1))$$

are both isomorphisms and the cycle map

$$\mathrm{H}^{2,1}_{\mathrm{M}}(\mathcal{X} \times_{\mathcal{Y}} (\mathrm{U} \cap \mathrm{V}), \mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}^2_{\mathrm{et}}(\mathcal{X} \times_{\mathcal{Y}} (\mathrm{U} \cap \mathrm{V}), \mu_{\ell^{\mathrm{n}}}(1))$$

is a monomorphism. Then the cycle map

$$\mathrm{H}^{2,1}_{\mathrm{M}}(\mathcal{X} \times_{\mathcal{Y}} (\mathrm{U} \cup \mathrm{V}), \mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}^2_{\mathrm{et}}(\mathcal{X} \times_{\mathcal{Y}} (\mathrm{U} \cup \mathrm{V}), \mu_{\ell^{\mathrm{n}}}(1))$$

is an isomorphism.

Proof. For a subscheme W in Y, we will continue to let $\mathcal{X}_W = \mathcal{X} \times_{\mathcal{Y}} \mathrm{W}$. Now we consider the commutative diagram with exact rows:

$$\begin{array}{ccccc}
\mathrm{H}^{1,1}_{\mathrm{M}}(\mathcal{X}_{\mathrm{U}}, \mathbb{Z}/\ell^{\mathrm{n}}) \oplus \mathrm{H}^{1,1}_{\mathrm{M}}(\mathcal{X}_{\mathrm{V}}, \mathbb{Z}/\ell^{\mathrm{n}}) & \longrightarrow & \mathrm{H}^{1,1}_{\mathrm{M}}(\mathcal{X}_{\mathrm{U}\cap\mathrm{V}}, \mathbb{Z}/\ell^{\mathrm{n}}) & \longrightarrow & \mathrm{H}^{2,1}_{\mathrm{M}}(\mathcal{X}_{\mathrm{U}\cup\mathrm{V}}, \mathbb{Z}/\ell^{\mathrm{n}}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^{1}_{\mathrm{et}}(\mathcal{X}_{\mathrm{U}}, \mu_{\ell^{\mathrm{n}}}(1)) \oplus \mathrm{H}^{1}_{\mathrm{et}}(\mathcal{X}_{\mathrm{V}}, \mu_{\ell^{\mathrm{n}}}(1)) & \longrightarrow & \mathrm{H}^{1}_{\mathrm{et}}(\mathcal{X}_{\mathrm{U}\cap\mathrm{V}}, \mu_{\ell^{\mathrm{n}}}(1)) & \longrightarrow & \mathrm{H}^{2}_{\mathrm{et}}(\mathcal{X}_{\mathrm{U}\cup\mathrm{V}}, \mu_{\ell^{\mathrm{n}}}(1))
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & \mathrm{H}^{2,1}_{\mathrm{M}}(\mathcal{X}_{\mathrm{U}}, \mathbb{Z}/\ell^{\mathrm{n}}) \oplus \mathrm{H}^{2,1}_{\mathrm{M}}(\mathcal{X}_{\mathrm{V}}, \mathbb{Z}/\ell^{\mathrm{n}}) & \longrightarrow & \mathrm{H}^{2,1}_{\mathrm{M}}(\mathcal{X}_{\mathrm{U}\cap\mathrm{V}}, \mathbb{Z}/\ell^{\mathrm{n}}) \\
& \downarrow & & \downarrow \\
\longrightarrow & \mathrm{H}^{2}_{\mathrm{et}}(\mathcal{X}_{\mathrm{U}}, \mu_{\ell^{\mathrm{n}}}(1)) \oplus \mathrm{H}^{2,1}_{\mathrm{et}}(\mathcal{X}_{\mathrm{V}}, \mu_{\ell^{\mathrm{n}}}(1)) & \longrightarrow & \mathrm{H}^{2}_{\mathrm{et}}(\mathcal{X}_{\mathrm{U}\cap\mathrm{V}}, \mu_{\ell^{\mathrm{n}}}(1))
\end{array}$$

In view of the spectral sequence in (9.4) with $j = 1$, one may observe that the second vertical map is an isomorphism. Therefore, Lemma 2.1(ii) applies to prove the required map is an isomorphism □

Proposition 9.6. *Let $p : \mathcal{X} \to \mathcal{Y}$ denote a map of smooth schemes over k, satisfying the hypotheses of Lemma 9.5. We will further assume the following: let $\mathrm{U_i}, \mathrm{i} = 1, \cdots, \mathrm{n}$ denote open subsets of $\mathcal{Y}$, so that the hypotheses of Lemma 9.5 holds with* U, V *denoting any two of these open sets. Assume further that there exists an affine space $\mathbb{A}^{\mathrm{N}}$ so that each $\mathrm{U_i} \cong \mathbb{A}^{\mathrm{N}}$ and that each intersection $\mathrm{U_{i_1}} \cap \mathrm{U_{i_2}} \cong \mathbb{G}_{\mathrm{m}} \times \mathbb{A}^{\mathrm{N}-1}$. Then the following holds, where for a subscheme* W *in* Y*, we will let $\mathcal{X}_W = \mathcal{X} \times_{\mathcal{Y}} \mathrm{W}$, and cycl will denote the higher cycle map:*

(i) *$cycl : \mathrm{H}^{2,1}_{\mathrm{M}}(\mathcal{X}_{(\mathrm{U}_1 \cup \cdots \cup \mathrm{U}_{\mathrm{n}-1}) \cap \mathrm{U}_{\mathrm{n}}}, \mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}^2_{\mathrm{et}}(\mathcal{X}_{(\mathrm{U}_1 \cup \cdots \cup \mathrm{U}_{\mathrm{n}-1}) \cap \mathrm{U}_{\mathrm{n}}}, \mu_{\ell^{\mathrm{n}}}(1))$ is a monomorphism and*
(ii) *$cycl : \mathrm{H}^{2,1}_{\mathrm{M}}(\mathcal{X}_{(\mathrm{U}_1 \cup \cdots \cup \mathrm{U}_{\mathrm{n}-1}) \cup \mathrm{U}_{\mathrm{n}}}, \mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}^2_{\mathrm{et}}(\mathcal{X}_{(\mathrm{U}_1 \cup \cdots \cup \mathrm{U}_{\mathrm{n}-1}) \cup \mathrm{U}_{\mathrm{n}}}, \mu_{\ell^{\mathrm{n}}}(1))$ is an isomorphism.*

Proof. We will prove these using ascending induction on n. We will first consider (i). Observe that the case $n = 2$ is handled by Lemma 9.7.

Assume next that (i) holds when U_i, $i = 1, \cdots, n$ are any open subsets of $\mathcal{Y}$ satisfying the hypotheses. Let U_i, $i = 1, \cdots, n, n+1$ be open subsets satisfying the hypotheses. Let $W_1 = (U_1 \cup \cdots \cup U_{n-1}) \cap U_{n+1}$ and let $W_2 = U_n \cap U_{n+1}$. Then we obtain the commutative diagram:

$$\begin{array}{ccccc}
H^{1,1}_M(\mathcal{X}_{W_1}, \mathbb{Z}/\ell^n) \oplus H^{1,1}_M(\mathcal{X}_{W_2}, \mathbb{Z}/\ell^n) & \longrightarrow & H^{1,1}_M(\mathcal{X}_{W_1 \cap W_2}, \mathbb{Z}/\ell^n) & \longrightarrow & H^{2,1}_M(\mathcal{X}_{W_1 \cup W_2}, \mathbb{Z}/\ell^n) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{et}(\mathcal{X}_{W_1}, \mu_{\ell^n}(1)) \oplus H^1_{et}(\mathcal{X}_{W_2}, \mu_{\ell^n}(1)) & \longrightarrow & H^1_{et}(\mathcal{X}_{W_1 \cap W_2}, \mu_{\ell^n}(1)) & \longrightarrow & H^2_{et}(\mathcal{X}_{W_1 \cup W_2}, \mu_{\ell^n}(1))
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & H^{2,1}_M(\mathcal{X}_{W_1}, \mathbb{Z}/\ell^n) \oplus H^{2,1}_M(\mathcal{X}_{W_2}, \mathbb{Z}/\ell^n) & \longrightarrow & H^{2,1}_M(\mathcal{X}_{W_1 \cap W_2}, \mathbb{Z}/\ell^n) \\
 & \downarrow & & \downarrow \\
\longrightarrow & H^2_{et}(\mathcal{X}_{W_1}, \mu_{\ell^n}(1)) \oplus H^{2,1}_{et}(\mathcal{X}_{W_2}, \mu_{\ell^n}(1)) & \longrightarrow & H^2_{et}(\mathcal{X}_{W_1 \cap W_2}, \mu_{\ell^n}(1))
\end{array}$$

Then the inductive assumption, together with Lemma 9.7 show the map $H^{2,1}_M(\mathcal{X}_{W_1}, \mathbb{Z}/\ell^n) \to H^2_{et}(\mathcal{X}_{W_1}, \mu_{\ell^n}(1))$ is a monomorphism while Lemma 9.7 shows the map $H^{2,1}_M(\mathcal{X}_{W_2}, \mathbb{Z}/\ell^n) \to H^2_{et}(\mathcal{X}_{W_2}, \mu_{\ell^n}(1))$ is a monomorphism. Observe that $W_1 \cup W_2 = (U_1 \cup \cdots \cup U_n) \cap U_{n+1}$. In view of the spectral sequence in (9.4) with $j = 1$, one may observe that the first two vertical maps are isomorphisms. Therefore, now an application of Lemma 2.1(i) then shows the cycle map $H^{2,1}_M(\mathcal{X}_{W_1 \cup W_2}, \mathbb{Z}/\ell^n) \to H^2_{et}(\mathcal{X}_{W_1 \cup W_2}, \mu_{\ell^n}(1))$ is a monomorphism, thereby completing the proof of (i).

At this point (ii) follows readily from Lemma 9.5 by taking $U = U_1 \cup \cdots \cup U_n$ and $V = U_{n+1}$ there. Now observe that $U \cap V = (U_1 \cup \cdots \cup U_n) \cap U_{n+1}$. (i) proved above shows that the cycle map

$$H^{2,1}_M(\mathcal{X} \times_{\mathcal{Y}} (U \cap V), \mathbb{Z}/\ell^n) \to H^2_{et}(\mathcal{X} \times_{\mathcal{Y}} (U \cap V), \mu_{\ell^n}(1))$$

is a monomorphism. The inductive assumption now shows that the cycle map

$$H^{2,1}_M(\mathcal{X} \times_{\mathcal{Y}} U, \mathbb{Z}/\ell^n) \to H^2_{et}(\mathcal{X} \times_{\mathcal{Y}} U, \mu_{\ell^n}(1))$$

is an isomorphism. Therefore, the hypotheses of Lemma 9.5 are satisfied, so that Lemma 9.5 applies to complete the proof of (ii). □

Lemma 9.7. *Assume that* X *is a smooth scheme so that the cycle map*

$$cycl : H^{i,1}_M(X, \mathbb{Z}/\ell^n) \to H^i_{et}(X, \mu_{\ell^n}(1))$$

is an isomorphism for all $0 \le i \le 2$. *Then the induced cycle map* $H^{i,1}_M(X \times \mathbb{G}_m, \mathbb{Z}/\ell^n) \to H^i_{et}(X \times \mathbb{G}_m, \mu_{\ell^n}(1))$ *is injective for for all* $0 \le i \le 2$.

Proof. In view of the observation (9.4) above, the above cycle map is an isomorphism for $i = 0$ or $i = 1$. Therefore, it suffices to consider the case $i = 2$. This follows from the commutative diagram of localization sequences:

$$\begin{array}{ccccc}
H^{2,1}_{X \times \{0\},M}(X \times \mathbb{A}^1, \mathbb{Z}/\ell^n) & \longrightarrow & H^{2,1}_M(X \times \mathbb{A}^1, \mathbb{Z}/\ell^n) & \longrightarrow & H^{2,1}_M(X \times \mathbb{G}_m, \mathbb{Z}/\ell^n) \\
\downarrow & & \downarrow & & \downarrow \\
H^2_{X \times \{0\},et}(X \times \mathbb{A}^1, \mu_{\ell^n}(1)) & \longrightarrow & H^2_{et}(X \times \mathbb{A}^1, \mu_{\ell^n}(1)) & \longrightarrow & H^2_{et}(X \times \mathbb{G}_m, \mu_{\ell^n}(1))
\end{array}$$

$$\begin{array}{ccccc} \longrightarrow & \mathrm{H}^{3,1}_{\mathrm{X}\times\{0\},\mathrm{M}}(\mathrm{X}\times\mathbb{A}^1,\mathbb{Z}/\ell^{\mathrm{n}}) & \longrightarrow & \mathrm{H}^{3,1}_{\mathrm{M}}(\mathrm{X}\times\mathbb{A}^1,\mathbb{Z}/\ell^{\mathrm{n}}) \\ & \downarrow & & \downarrow \\ \longrightarrow & \mathrm{H}^{3}_{\mathrm{X}\times\{0\},\mathrm{et}}(\mathrm{X}\times\mathbb{A}^1,\mu_{\ell^{\mathrm{n}}}(1)) & \longrightarrow & \mathrm{H}^{3}_{\mathrm{et}}(\mathrm{X}\times\mathbb{A}^1,\mu_{\ell^{\mathrm{n}}}(1)) \end{array}$$

The map $\mathrm{H}^{3,1}_{\mathrm{X}\times\{0\},\mathrm{M}}(\mathrm{X}\times\mathbb{A}^1,\mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}^{3}_{\mathrm{X}\times\{0\},\mathrm{et}}(\mathrm{X}\times\mathbb{A}^1,\mu_{\ell^{\mathrm{n}}}(1))$ identifies with the map

$$\mathrm{H}^{1,0}_{\mathrm{M}}(\mathrm{X}\times\{0\},\mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}^{1}_{\mathrm{et}}(\mathrm{X}\times\{0\},\mu_{\ell^{\mathrm{n}}}(0))$$

and $\mathrm{H}^{1,0}_{\mathrm{M}}(\mathrm{X}\times\{0\},\mathbb{Z}/\ell^{\mathrm{n}}) \cong \mathrm{CH}^0(\mathrm{X}\times\{0\},\mathrm{Z}/\ell^{\mathrm{n}};-1) \cong 0$. Therefore this map is clearly injective. The map $\mathrm{H}^{2,1}_{\mathrm{X}\times\{0\},\mathrm{M}}(\mathrm{X}\times\mathbb{A}^1,\mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}^{2}_{\mathrm{X}\times\{0\},\mathrm{et}}(\mathrm{X}\times\mathbb{A}^1,\mu_{\ell^{\mathrm{n}}}(1))$ identifies with the map

$$\mathrm{H}^{0,0}_{\mathrm{M}}(\mathrm{X}\times\{0\},\mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}^{0}_{\mathrm{et}}(\mathrm{X}\times\{0\},\mu_{\ell^{\mathrm{n}}}(0))$$

which is also an isomorphism. Now the required assertion follows from the following lemma. □

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Department of Mathematics, University of Western Ontario, London, Ontario, Canada
Email address: adhill3@uwo.ca

The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600113, India
Email address: jniyer@imsc.res.in

Department of Mathematics, The Ohio State University, Columbus, Ohio, 43210, USA
Email address: joshua.1@math.osu.edu