APPROXIMATION THEOREMS FOR CLASSIFYING STACKS OVER NUMBER FIELDS

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Abstract. Approximation theorems for algebraic stacks over a number field k are studied in this article. For G a connected linear algebraic group over a number field we prove strong approximation with Brauer-Manin obstruction for the classifying stack BG. This result answers a very concrete question, given G-torsors P_v over k_v, where v ranges over a finite number of places, when can you approximate the P_v by a G-torsor P defined over k.

1. Introduction

This paper is motivated by the following question, let G be a connected linear algebraic group over a number field k and let $v_1, \dots, v_n$ be some non-archimedean places of k. Denote by k_{v_i} the completion of k at v_i. Fix G-torsors P_i over each k_{v_i}. Given this data, can one find a G-torsor P over k that specialises to each of the fixed torsors? We provide a condition (vanishing of the Brauer-Manin obstruction) on the P_i in order to guarantee the existence of a P that is v_i-adically close to P_i. It is worth remarking here that Lang's theorem implies that every G-torsor over $\mathcal{O}_{v_i}$ is trivial where $\mathcal{O}_{v_i}$ is the completion of the ring of integers at the place non-archimedean place v_i so that this question is a version of the strong approximation property for the classifying stack of G-torsors. Note that for an algebraic group, being connected and geometrically connected are the same, as the identity component will be preserved by Galois actions.

We let BG be the classifying stack of G-torsors and let $\mathbf{A}_k$ be the adele ring of k. In section 2, we equip the adelic points of BG with the structure of a topological space. The essential idea for this construction can be found in [10]. There is a map $BG(k) \to BG(\mathbf{A}_k)$ induced from the natural embedding $k \to \mathbf{A}_k$. Note that this map need not be surjective nor injective, as discussed in the next paragraph. Our results compute the closure of the image of $BG(k)$ inside $BG(\mathbf{A}_k)$, see 5.5. A variant of this theorem that applies to quotient stacks is produced in 5.4. The main source of examples to which this theorem applies are generated by quotients of groupic varieties, see [8] and the discussion after 5.4. Note however, theorem 5.5 is not a corollary of 5.4.

The map $BG(k) \to BG(\mathbf{A}_k)$ has been studied in classically in a different guise, as a map on Galois cohomology. For PGL_n the map need not be surjective as class field theory describes the Brauer group of a number field in terms of local Brauer groups and hence puts a restriction upon which PGL_n-torsors can lift. The non-injectivity of the map is rather subtle and it often is injective. The injectivity is known as the Hasse principle and it may fail for some groups. For a general discussion see [33, pg. 285].

Given a scheme X of finite type over k, it is well known how to equip $X(k_v)$ with the structure of a topological space, see [12]. The insight of [10] is if $\mathfrak{X}$ is a finite type algebraic stack and $X \to \mathfrak{X}$ a smooth surjective presentation then $\mathfrak{X}(k_v)$ should inherit the quotient topology from $X(k_v)$ if the properties of the topologization on schemes are to extend to algebraic stacks. The paper [10] provides a topologization on schemes over fields or rings more general than just a number field or a global field. Unfortunately, in topologizing the adelic points of $\mathfrak{X}$ the paper [10] contains an error, c.f. [10, 5.0.3], as projective modules over the adele ring of k need not be trivial. Indeed over a product of rings it is straightforward to construct non-trivial examples of projective modules. Only those of locally constant rank are trivial, see 2.2. In §2 we develop a workaround for a broad class of algebraic stacks that includes all algebraic stacks of interest to us. It should be noted however, the essential idea in this workaround is based on the paper [10].

The Brauer group $\mathrm{Br}(\mathfrak{X})$ produces an obstruction to the closure of $\mathfrak{X}(k)$ being all of $\mathfrak{X}(\mathbf{A}_k)$. In section 4, we construct the Brauer-Manin pairing for algebraic stacks:

$$< -, - >: \mathfrak{X}(\mathbf{A}_k) \times \mathrm{Br}(\mathfrak{X}) \to \mathbb{Q}/\mathbb{Z}$$

and prove it's main properties needed for this paper. The pairing vanishes on k-rational points of $\mathfrak{X}(k)$ and the vanishing locus is a closed subset of $\mathfrak{X}(\mathbf{A}_k)$. In the case of varieties and schemes this obstruction is well-known, see [36], [13], [22], [29], [11] and [7].

Our approach to questions of strong approximation for algebraic stacks is via an idea borrowed from [4], see also [17, remark 2.11]. Given a quotient stack $[X/G]$ where G is a linear algebraic group over k then we can always write it as a quotient stack of the form $[X'/H]$ where H is a special group. Indeed, we can fix a faithful representation $G \hookrightarrow \mathrm{GL}(V)$ or a representation $G \hookrightarrow \mathrm{SL}(V)$ and write

$$[X \times_G \mathrm{GL}(V)/\mathrm{GL}(V)] \cong [X/G] \cong [X \times_G \mathrm{SL}(V)/\mathrm{SL}(V)].$$

A linear algebraic group G always admits a faithful representation into $\mathrm{SL}(V)$, see the discussion at the start of §5.2. After fixing $G \hookrightarrow \mathrm{SL}(V)$ we are able to produce a stream lined approach to strong approximation as the pullback map

$$\mathrm{Br}([X/G]) \to \mathrm{Br}(X \times_G \mathrm{SL}(V))$$

can be shown to be an isomorphism, see 5.3. This reduces the question of strong approximation to one about approximation on $X \times_G \mathrm{SL}(V)$. When $X = \mathrm{Spec}(k)$ and G is a connected linear algebraic group this has been considered in [7] and [11]. In these papers, a kind of equivariant version of strong approximation, weaker than strong approximation as we have defined it, is proved. This result is sufficient for our purposes. These works are a culmination of a long development of ideas, see also [22] and [6]. We borrow their ideas to settle strong approximation for classifying stacks BG where G is a connected linear algebraic group. Further examples of algebraic stacks to which strong approximation applies can be generated by considering certain quotients of groupic varieties, see [8] and the discussion after 5.4.

In 5.3 we produce a simple example to show that the connectedness assumption is needed. This is not surprising. Indeed for homogeneous varieties with disconnected stabilisers, a more powerful invariant, the étale Brauer-Manin obstruction is needed to produce strong approximation type theorems, see [14] and [3]. The étale Brauer-Manin obstruction for algebraic stacks will be considered in future work. The paper [7] also considers strong approximation homogeneous spaces of abelian varieties. Our methods do not apply in this situation. Indeed, to topologize $BE(\mathbf{A}_k)$, where E is an abelian variety, our procedure does not work as E is not a subgroup of $\mathrm{GL}(V)$. To topologize this space one would need to follow the procedure of constructing the adelic topology from the p-adic topology as in the case of ordinary varieties.

In section 2 of the paper we introduce the topology on the adelic points of a algebraic stack and prove the main properties of this topology. The previously pointed out error in [10] is corrected here. However, many of the other ideas presented in this section are borrowed from *op. cit.*

Section 3 discusses the Brauer group of an algebraic stack is defined, in this paper to be the cohomological Brauer group. We briefly discuss how to spread out a Brauer class in this section. As this result is mostly a standard extension of a known result for schemes we only sketch the required argument.

In section 4, we construct the Brauer-Manin pairing on an algebraic stack and prove it's main properties. In particular, it is shown that the locus inside the adelic points of an algebraic stack consisting of points orthogonal to the Brauer-Manin pairing form a closed subset that contains the k-rational points. This produces an obstruction to strong approximation for the entire stack. In the final subsection, we recall the definition of strong and weak approximation with respect to the Brauer-Manin pairing. As a prelude to the arguments to come, we show how [6] implies weak approximation for BG, see 4.8.

Section 5 proves the main result on strong approximation for BG after some preliminary lemma's. The section concludes with an example that shows that the connectedness assumptions are needed.

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Table 1: Notation

$[X/G]$	If G is a group scheme acting on the scheme X then this is the corresponding quotient stack.
k	A number field.
$\mathcal{O}_k$	The ring of integers in k.
Ω_k	The set of places of k.
Ω_k^{fin}	The set of non-archimedean places of k.
Ω_k^{∞}	The set of archimedean places of k.
k_v	The completion of k at the valuation v.
$\mathcal{O}_v$	The valuation subring of k_v.
k_S	For some finite subset S of Ω_k this is the ring $\prod_{v\in S} k_v$.
k^T	For some finite subset T of Ω_k this is the ring $\prod_{v\in T} k_v \times \prod_{v\notin T} \mathcal{O}_v$
$(k^T)_{\not S}$	For some finite subsets S and T of Ω_k this is the projection of k^T into $\prod_{v\notin S} k_v$
$\mathbf{A}_k$	The adele ring of k
$\mathbf{A}_{\not S,k}$	For a finite subset $S \subseteq \Omega_k$, this is the image of $\mathbf{A}_k$ under projection into the product $\prod_{v\notin S} k_v$.

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Table 1: Notation (Continued)

$\mathrm{Br}(\mathfrak{X})$	The Brauer group of a stack or scheme.
$\mathrm{Br}_1(\mathfrak{X})$	The subgroup of the Brauer group trivialised by passage to an algebraic closure. Precisely $\ker(\mathrm{Br}(\mathfrak{X}) \to \mathrm{Br}(\mathfrak{X} \otimes \overline{k}))$.
$\mathrm{Br}_a(\mathfrak{X})$	The quotient of the $\mathrm{Br}_1(\mathfrak{X})$ by the image of $\mathrm{Br}(k)$.
G	A linear algebraic group over k. Often it will be assumed to be connected.

2. The topological structure of the adelic points of an algebraic stack

Given an algebraic stack X over a global field, the paper [10] associates to it a topological space with underlying set the adelic points of X. Unfortunately, the construction has an error at a critical point, see [10, Remark 12.0.6]. Indeed given a product of rings then there are non-free finite rank projective modules over such a product so such a product cannot be sufficiently disconnected in the terminology of [10, Definition 5.0.3].

In this paper we will sidestep this issue, and combine the basic idea of [10] with an idea from [4] to construct a topological space associated to a suitable category of algebraic stacks over a number field. This category will contain all stacks that we are interested in for this paper.

2.1. A little lemma.

We will make extensive use of the following.

Lemma 2.1. *Let X/k be a scheme of finite type and $H \hookrightarrow G$ an closed inclusion of linear algebraic groups over k. Suppose that H acts on X. Then we have an isomorphism*

$$[X/H] \cong [X \times_H G/G]$$

of quotient stacks.

Proof. The proof follows easily from definitions. See also [17, 2.11]. □

2.2. Vector bundles over the adeles.

Proposition 2.2. *Let V be a k-vector space of finite dimension and let $S \subset \Omega_k$ be some finite subset. Every* $\mathrm{GL}(V)$*-torsor over* $\mathrm{Spec}(\mathbf{A}_{\not S,k})$ *is trivial.*

Proof. Giving a principal $\mathrm{GL}(V)$-bundle over $\mathrm{Spec}(\mathbf{A}_{\not S,k})$, for some finite subset $S \subseteq \Omega_k$, is equivalent to giving a finitely generated projective module P over $\mathbf{A}_{\not S,k}$ of constant rank $\dim V$. We can write $P \oplus Q \cong \mathbf{A}^n_{\not S,k}$ for some projective module Q. It follows that P is the kernel of

$$p : \mathbf{A}^n_{\not S,k} \twoheadrightarrow Q \hookrightarrow \mathbf{A}^n_{\not S,k}$$

with $p^2 = p$. We write $[p]$ for the matrix representing p. It is defined over $(k^T)_{\not S}$ for some finite subset $T \subset \Omega_k$ so that

$$[p] \in \mathrm{Mat}_{n\times n}(k^T_{\not S}).$$

For each $v \in \Omega_k \setminus S$ we obtain a matrix

$$[p]_v \in \mathrm{Mat}_{n\times n}(k_v)$$

which will have coefficients in $\mathcal{O}_v$ whenever $v \notin T$. Now $\ker([p]_v)$ is a projective module as p is idempotent, and it has rank $\dim V$ as P has constant rank. As $\mathcal{O}_v$ is local we can find isomorphisms

$$\ker([p]_v) \cong \begin{cases} k_v^{\dim V} & \text{if } v \in T \\ \mathcal{O}_v^{\dim V} & \text{otherwise.} \end{cases}$$

It follows that $\ker([p])$ is a trivial projective k^T-module. As P is obtained from it via base extension, it is trivial also. $\square$

Corollary 2.3. *Every* $\mathrm{SL}(V)$*-torsor over* $\mathrm{Spec}(\mathbf{A}_{\not{\emptyset},k})$ *is trivial.*

Proof. A $\mathrm{SL}(V)$-torsor is a $\mathrm{GL}(V)$-torsor equipped with a reduction of structure group to $\mathrm{SL}(V)$. The reduction of structure group amounts to a trivialization of the top exterior power of the projective $\mathbf{A}_{\not{\emptyset},k}$-module associated to the $\mathrm{GL}(V)$-torsor. We have seen that the projective module is trivial so the reduction amounts to a choice of

$$w \in \bigwedge^n \mathbf{A}^n_{\not{\emptyset},k}$$

that induces an isomorphism

$$\mathbf{A}_{\not{\emptyset},k} \xrightarrow{\sim} \bigwedge^n \mathbf{A}^n_{\not{\emptyset},k}\,.$$

Any two choices of w can be identified via an automorphism of $\mathbf{A}^n_{\not{\emptyset},k}$ and hence the associated $\mathrm{SL}(V)$-torsor is trivial. To make this explicit, one can choose a basis, say e_i, for the free module $\mathbf{A}^n_{\not{\emptyset},k}$. Then

$$w = \lambda e_1 \wedge \ldots \wedge e_n$$

where $\lambda \in \mathbf{A}^\times_{\mathbf{C},k}$ is an idele. Then one considers the new basis

$$\lambda e_1, e_2, \ldots, e_n$$

and the associated change of basis automorphism of $\mathbf{A}^n_{\not{\emptyset},k}$. $\square$

2.3. **Topologies on schemes and algebraic spaces over topological rings.** Given a finite type scheme X over a local field k_v, the collection of k_v-points of X has the structure of a topological space. For a finite type scheme X over k, it's adelic points, $X(\mathbf{A}_k)$ also has the structure of a topological space.

If R is a Hausdorff topological ring. Denote by $\mathrm{Aff}_R^{\text{fin.type}}$ the category of finite type affine schemes over R and let Top be the category of topological spaces. Then there is a functor

$$F : \mathrm{Aff}_R^{\text{fin.type}} \longrightarrow \mathrm{Top}$$

that is essentially unique. We refer the reader to [12, Proposition 2.1] for a construction of this functor, it's basic properties and the precise meaning of "essentially unique".

Remark 2.4. Using this results, the collection of units $R^\times$ obtains a topology from the closed embedding $\mathbb{G}_m \hookrightarrow \mathbb{A}^2$ as the subscheme $V(xy-1)$. There are two potential topologies on $R^\times$, one from the proposition and the other from the subspace topology coming from the topology on R. These two topologies need not be the same. In fact, for the main example that we will be interested in, the ring of adeles $R = \mathbf{A}_k$ they are different.

Definition 2.5. Let R be a Hausdorff topological ring. We say that R has *continuous inversion* if the two topologies on $R^\times$ are the same.

The fields k_v have continuous inversion.

As a consequence the inverse map on $R^\times$ will be continuous with respect to the subspace topology for a ring with continuous inversion. When our ring has continuous inversion the functor above extends to functor on finite type schemes, see [12, Proposition 3.1].

When R is a completion of k or a completion of $\mathcal{O}_k$ we can say slightly more.

Proposition 2.6. *Let $f : X \to Y$ be a smooth morphism between finite type $\mathcal{O}_v$-schemes. Let $R = k_v$ or $\mathcal{O}_v$. Then*

$$f : X(R) \to Y(R)$$

is an open morphism of topological spaces. Hence, the subspace topology on $f(X(R))$ is quotient topology induced from the topology on $X(R)$.

Proof. Being an open map is local on the domain. In view of the previous proposition we may replace X by an open subscheme. After shrinking Y, we can find an open affine cover V_i of X and local factorizations of f as

$$f|_{V^i} : V_i \to \mathbb{A}^n \times Y \overset{\text{proj}}{\to} Y$$

where the first map is etale. From [12, Lemma 5.3] the first map induces a local homeomorphism on R-points. As the topology on the product, is the product topology, the projection is also an open morphism. ☐

The above observation is the starting point in [10] in order to topologize points of stacks.

To define a topology on the adelic points of a scheme we need a different approach as the adele ring does not have continuous inversion. Fix a finite subset S of Ω_k containing all infinite places. We denote by $\mathcal{O}_{k,S}$ the intersection of k with k^S. Alternatively,

$$\mathcal{O}_{k,S} = \{x \in k \mid v(x) \geq 0 \ \forall v \notin S\}.$$

There is an inclusion

$$\mathcal{O}_{k,S} \hookrightarrow k^S.$$

Theorem 2.7. *Let $X \in \mathrm{Sch}^{\text{fin.type}}_{\mathcal{O}_{k,S}}$. Then the natural map*

$$X(k^S) \to \prod_{v \in S} X(k_v) \times \prod_{v \notin S} X(\mathcal{O}_v)$$

is a bijection. If we use this to equip $X(k^S)$ with a topology by using the product topology on the right hand side and the above discussion then one obtains a functor to topological spaces. This functor coincides with the functor in the above discussion when restricted to affine schemes. For $S \subseteq S'$ the natural map $X(k^S) \hookrightarrow X(k^{S'})$ open inclusion for separated schemes.

Proof. See [12, 3.6]. ☐

For a scheme X of finite type over k one obtains a topological space structure on it's adelic points by first spreading X out to a finite type scheme over $\mathcal{O}_{k,S}$ for some subset $S \subset \Omega_k$ and then taking a direct limit of the topological spaces $X(k^{S'})$ for $S' \supseteq S$. We refer the reader to [12, Theorem 3.6] and the surrounding discussion for a complete argument. Spreading out schemes is discussed in [34, 3.2].

The analogue of 2.6 also holds true for adelic points.

Proposition 2.8. *Let $f : X \to Y$ be a smooth morphism between finite type $\mathcal{O}_{k,S}$-schemes. Let $R = k^S$. Suppose further that the induced map*

$$f : X(\mathcal{O}_v) \to Y(\mathcal{O}_v)$$

is surjective for almost all $v \notin S$. Then

$$f : X(R) \to Y(R)$$

is an open morphism of topological spaces. Hence, the subspace topology on $f(X(R))$ is quotient topology induced from the topology on $X(R)$.

Proof. Compare [10, 4.0.5]. A basic open set of $X(R)$ is of the form,

$$\prod_{s \in S} U_s \times \prod_{t \in T} V_t \times \prod_{t \notin T} X(\mathcal{O}_t)$$

where $T \subseteq \Omega_k$ is a finite set disjoint from S, $V_t \subseteq X(\mathcal{O}_t)$ is open, and $U_s \subseteq X(k_s)$ is open. The result follows from 2.6 and the definition of the product topology. □

If the base topological ring is a complete valued field, denoted k_v, then the above constructions can be extended to algebraic spaces.

Theorem 2.9. *Denote by $\mathrm{Alg.Sp}_{k_v}$ the category of quasi-separated algebraic spaces locally of finite type over k_v. Then the above functor can be extended to a functor*

$$\mathrm{Alg.Sp}_{k_v} \to \mathrm{Top}.$$

This functor preserves fibred products, open and closed immersions. Étale morphism are sent to local homeomorphisms.

Proof. This is in [12, §5]. The essential idea is given an algebraic space X and a subset $S \subseteq X(k_v)$, we will say that S is open if and only if $f^{-1}(S)$ is open for every étale morphism $f : U \to X$ where U is a finite type scheme. We have previously topologized $U(k)$. □

Proposition 2.10. *Let $f : X \to Y$ be a smooth morphism between finite type $\mathcal{O}_v$-algebraic spaces. Let $R = k_v$ or $\mathcal{O}_v$. Then*

$$f : X(R) \to Y(R)$$

is an open map. Hence the subspace topology on $f(X(R))$ is quotient topology induced from the topology on $X(R)$.

Proof. The proof follows in the same way as for schemes, see 2.6. Note that for an étale morphism between algebraic spaces, the assertion is proved in [12, 5.4]. One can prove the required factorisation for smooth maps of algebraic spaces by construct the factorisation on an atlas. □

Given a finite type algebraic space over k, one can spread it out. We will briefly describe this construction in §4. One can now construct a topological space structure on the adelic points of an algebraic space, mimicking the construction for schemes. This is carried out in [12, pg. 90] which we refer the reader to for further details. Let us just recall a few key components of the construction. It X is an algebraic space over $\mathcal{O}_{k,S}$ of finite type then there is a natural bijection

$$X(k^S) \longrightarrow \prod_{v\in S} X(k_v) \times \prod_{v\notin S} X(\mathcal{O}_v).$$

This bijection is used to equip the right hand side with the product topology. It follows that if $S \subseteq T$ then the natural map

$$X(k^S) \hookrightarrow X(k^T)$$

is an inclusion of an open set. Then the topology on

$$X(\mathbf{A}_k) = \varinjlim_{S\subseteq\Omega_k} X(k_S)$$

is the colimit topology.

The smooth quotient property holds also for algebraic spaces.

Proposition 2.11. *Let $f : X \to Y$ be a smooth morphism between finite type $\mathcal{O}_{k,S}$-algebraic spaces. Suppose further that the induced map*

$$f : X(\mathcal{O}_v) \to Y(\mathcal{O}_v)$$

is surjective for almost all $v \notin S$. Then

$$f : X(k^S) \to Y(k^S)$$

is an open morphism of topological spaces. Hence, the subspace topology on $f(X(k^S))$ is quotient topology induced from the topology on $Y(k^S)$.

Note that we will briefly discuss how to spread out finite type algebraic spaces in §4 so that the last part of the proposition makes sense.

Proof. This is carried out in [10, §8] but it readily follows from the discussion above. □

2.4. **Adelic topologies on algebraic stacks.** In this subsection we wish to define a topological space structure on $\mathfrak{X}(\mathbf{A}_k)$ for a certain class of algebraic stacks over k. We will discuss how to spread out finite type algebraic stacks in 4.2.

Definition 2.12. Let $\mathfrak{X}$ be an algebraic stack of finite type over k_v. We say that $\mathfrak{X}$ is k_v-liftable, or just liftable when the context is clear if there is a presentation $P \to \mathfrak{X}$ where P is a finite type algebraic space over k_v such that the induced map

$$P(k_v) \to \mathfrak{X}(k_v)$$

is surjective. In this situation, we call P a lifting presentation.

Remark 2.13. In the situation of the definition, if $\mathfrak{X}$ has affine stabilizers then it is k_v-liftable by [15, Theorem 1.2].

Definition 2.14. Let $S \subset \Omega_k$ be a finite set containing all infinite places.

Let $\mathfrak{X}$ be an algebraic stack over $\mathcal{O}_{k,S}$. We say that $\mathfrak{X}$ is *S-liftable* if there is a single presentation $P \to \mathfrak{X}$ with P a finite type $\mathcal{O}_{k,S}$-algebraic space such that

(1) every $s \in \mathfrak{X}(k^T)$ lifts to P where T is a finite subset of Ω_k containing S,
(2) and the induced map $P(\mathcal{O}_v) \to \mathfrak{X}(\mathcal{O}_v)$ is surjective for all but finitely many $v \notin S$.

In this situation, we will call $P \to \mathfrak{X}$ a S-lifting presentation.

Given an algebraic stack over k of finite type, we will discuss in 4.2 how one can spread it out to a stack over $\mathcal{O}_{k,S}$ for some subset S of Ω_k.

Examples of S-liftable stacks abound. Indeed any quotient stack $X = [P/G]$ where P is a finite type $\mathcal{O}_{k,S}$-scheme and G is a linear algebraic group over $\mathcal{O}_{k,S}$ is S-liftable. To see this, choose a faithful representation $G \hookrightarrow \mathrm{GL}_n$ and observe that $[P/G] = [P \times_G \mathrm{GL}_n \,/\, \mathrm{GL}_n]$, see 2.1. Then the presentation $P \times_G \mathrm{GL}_n \to X$ has the required property as every GL_n-torsor over k^S is trivial, see 2.2 and it's proof. Further, as the ring $\mathcal{O}_v$ is local, the second condition also holds. In a similar way examples of k_v-liftable stacks exist.

Remark 2.15. For a stack $\mathfrak{X}$ over $\mathcal{O}_v$ satisfying one of the following conditions

(1) $\mathfrak{X}$ has affine stabilizers,
(2) $\mathfrak{X}$ has quasi-affine diagonal,
(3) $\mathfrak{X}$ is Deligne-Mumford

then the second condition of the previous definition is automatic by [21, Corollary 1.5] and formal smoothness.

If $\mathfrak{X}$ is a liftable algebraic stack over k_v then we topologize $\mathfrak{X}(k_v)$ in the following way. First fix a lifting presentation $P \to \mathfrak{X}$ so that $P(k_v) \to \mathfrak{X}(k_v)$ surjects. Then equip $\mathfrak{X}(k_v)$ with the quotient topology. Note that if $X \to Y$ is a morphism of schemes of finite type which is smooth and surjective on k_v-points then the topology on $Y(k_v)$ is necessarily the quotient topology of that on $X(k_v)$ by 2.6.

Similarly, if $\mathfrak{X}$ is a S-liftable stack we equip $\mathfrak{X}(k^S)$ with the quotient topology via the morphism

$$P(k^S) \to \mathfrak{X}(k^S).$$

Proposition 2.16. *The above definitions do not depend on the choice of lifting presentation. It is functorial for representable morphisms.*

Proof. The proof for S-liftable stacks will be given. The other case is similar, and easier.

Suppose that we have two lifting presentations $P_i \to \mathfrak{X}$ then the fibered product $P_1 \times_{\mathfrak{X}} P_2$ is also a lifting presentation. The morphism $P_1 \times_{\mathfrak{X}} P_2 \to P_i$ is smooth and surjective and satisfies the hypothesis of 2.11. Hence the continuous map

$$(P_1 \times_{\mathfrak{X}} P_2)(k_v) \to P_i(k_v)$$

is a quotient map of topological spaces. The result follows by composing quotients.

Consider a representable morphism $\mathfrak{X} \to \mathfrak{Y}$ of liftable stacks. Then given a lifting presentation $Y \to \mathfrak{Y}$ the fibered product $Y \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{X}$ is a lifting presentation. The result follows from the independence of the topology from the presentation. □

Lemma 2.17. *Let $\mathfrak{X}$ be a S-liftable stack over $\mathcal{O}_{k,S}$. Suppose that $P \to \mathfrak{X}$ is a lifting presentation with P a separated algebraic space. Then for every finite subset $T \subset \Omega_k$ containing S, the stack is $\mathfrak{X} \otimes_{\mathcal{O}_{k,S}} \mathcal{O}_{k,T}$ is T-liftable. We also have that the natural map*

$$\mathfrak{X}(k^S) \to \mathfrak{X}(k^T)$$

is an open morphism.

Proof. The first assertion is clear. The analogous result holds for algebraic spaces, see [12, §5] so that

$$P(k^S) \subseteq P(k^T)$$

is open. The result now follows from the fact that the topology on $\mathfrak{X}(k^T)$ is the quotient topology inherited from $P(k^T)$. $\square$

Given a S-liftable stack $\mathfrak{X}$ we define a topology on it's adelic points by equipping

$$\mathfrak{X}(\mathbf{A}_k) = \varinjlim_{S \subseteq \Omega_k} \mathfrak{X}(k^S)$$

with the colimit topology in topological spaces. The colimit is over all finite subsets of Ω_k. In subsection 4.2 we will discuss spreading out stacks over k to stacks over $\mathcal{O}_{k,S}$.

3. The Brauer group of an algebraic stack

3.1. **The big étale site.** If $\mathfrak{X}$ is a an algebraic stack then $(\mathrm{Sch}/\mathfrak{X})_{\acute{e}t}$ will denote the big étale site on $\mathfrak{X}$. This is the category with objects morphisms $X \to \mathfrak{X}$ where X is a scheme and morphisms are commuting triangles. The covers are étale covers of schemes. This site is functorial for morphisms of algebraic stacks, see [37, Tag 06NW], unlike other commonly used sites such as the lisse-étale site, see [31, 3.3], [5, 4.42] or [37, Tag 07BF].

For every scheme X and every morphism $f : X \to \mathfrak{X}$ there is an inclusion functor $(\mathrm{Sch}/X)_{\acute{e}t} \to (\mathrm{Sch}/\mathfrak{X})_{\acute{e}t}$ obtained by composing with f.

If F is a sheaf on $(\mathrm{Sch}/\mathfrak{X})_{\acute{e}t}$ then for every morphism $f : X \to \mathfrak{X}$ we obtain a restricted sheaf on $(\mathrm{Sch}/X)_{\acute{e}t}$ that we denote by $F|_X$ when the morphism f is clear from the context.

Theorem 3.1. *Let R be an integral domain with field of fractions K. Let $\tilde{\mathfrak{X}}$ be an algebraic stack over R of finite type so that it has a presentation $X \to \tilde{\mathfrak{X}}$ so that X is an R-scheme of finite type. Let $\sigma \in H^2(\tilde{\mathfrak{X}}_K, \mathbb{G}_{\mathrm{m}})$ then there is an $a \in R$ so that σ lifts to $\tilde{\mathfrak{X}}_a$.*

Proof. We start by remarking that this result is well-known for schemes and can easily be extended to algebraic spaces, for example by adapting the argument below to pass from schemes to algebraic spaces. Given this, we proceed to sketch the proof for an algebraic stack as the details are relatively standard. The argument for schemes can be found in [2, Expose V, 5.7-5.8].

Choose a presentation $X \to \tilde{\mathfrak{X}}$ as in the statement of the theorem. Then the simplicial algebraic space $X_\bullet$ formed by taking the coskeleton of $u : X \to \tilde{\mathfrak{X}}$ has the same cohomology with coefficients in $u^{-1}F$ as the stack. This follows from [37, Tag 06XF]. The result follows from the known result for algebraic spaces. $\square$

3.2. **The Brauer group of an algebraic stack.** Let $\mathfrak{X}$ be an algebraic stack. We define its Brauer group to be

$$\mathrm{Br}(X) := H^2((\mathrm{Sch}/\mathfrak{X})_{\acute{e}t}, \mathbb{G}_{\mathrm{m}\mathfrak{X}})_{tors}.$$

One could potentially use the lisse-etale site to define this but this makes no difference, see [23, A.1].

If $\mathfrak{X}$ is an algebraic stack over k, our number field we introduce some variations on the Brauer group that will play an important role later. We define

$$\mathrm{Br}_1(\mathfrak{X}) := \ker\left(\mathrm{Br}(\mathfrak{X}) \to \mathrm{Br}(\mathfrak{X} \otimes_k \overline{k})\right)$$

and

$$\mathrm{Br}_a(\mathfrak{X}) := \mathrm{Br}_1(\mathfrak{X}) / \left(p^* \,\mathrm{Br}(k)\right)$$

where $p : \mathfrak{X} \to \mathrm{Spec}(k)$ is the structure map.

These definitions are just adaptations to algebraic stacks of the corresponding constructions in [36].

4. The Brauer-Manin pairing for algebraic stacks

4.1. **Spreading out schemes, algebraic spaces and their morphisms.** We will follow the conventions of [37, Tag 0ELT] regarding algebraic spaces. An *algebraic space* X is a a sheaf on the fppf site of Sch such that $X \to X \times X$ is representable and there is a scheme U and a surjective étale morphism $U \to X$. Such data is equivalent to giving an étale equivalence relation on a scheme. In the present context, the equivalence relation is recovered as

$$U \times_X U \rightrightarrows U.$$

The scheme U is called a presentation for X. Some pertinent definitions will be recalled in its proof.

The following result is well-known and we record it here for future use.

Theorem 4.1. *Let R be an integral domain with field of fractions K.*

(1) If X is an algebraic space of finite type over K then there is open subscheme $V \subseteq$ $\mathrm{Spec}R$ and an algebraic space $\tilde{X} \to V$ whose generic fibre is X. Further, $\tilde{X}$ is of finite presentation over V.
(2) If $\tilde{X}$ and $\tilde{Y}$ are algebraic spaces of finite presentation over $\mathrm{Spec}R$ and $f : \tilde{X}_K \to \tilde{Y}_K$ is a morphism over their generic fibers then f can be lifted to a morphism $\tilde{f} : \tilde{X}_V \to \tilde{Y}_V$ for some open subscheme $V \subseteq \mathrm{Spec}R$.
(3) In the situation of the previous part, if f is smooth or étale then there is an open subscheme of $\mathrm{Spec}R$ over which the lift is smooth or étale.
(4) In the above situation suppose that we have finitely presented algebraic spaces $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ over S with generic fibers X, Y and Z. If $\tilde{f} : \tilde{X} \to \tilde{Y}$, $\tilde{g} : \tilde{Y} \to \tilde{Z}$ and $\tilde{h} : \tilde{X} \to \tilde{Z}$ are morphisms whose generic fibers satisfy $g \circ f = h$ then there is an open subscheme U of S with

$$\tilde{g} \circ \tilde{f} = \tilde{h}$$

over U.

Proof. For a scheme [34, 3.2.1] proves the first three parts. For a scheme the last part is an elementary result of commutative algebra.

We move on to the case of an algebraic space.

The fact that X is of finite type means that we can find a presentation $U \to X$ with U of finite type over K, see [37, Tag 03XE]. It follows that $U \times_X U$ is also of finite type. By the result for schemes, we can find an open subscheme $V \subseteq \mathrm{Spec}R$ and schemes of finite presentation $\tilde{Q}$ and $\tilde{U}$ that extend $U \times_X U$ and U respectively. By the same result, we can assume that the two projections $U \times_X U \to U$ extend to étale morphisms and the morphism $U \times_X U \to U \times_K U$ extends to a monomorphism

$$m : \tilde{Q} \to \tilde{U} \times_R \tilde{U}.$$

It is claimed that after further restricting V we can assume that this data forms an étale equivalence relation. For example, to check the first condition of being an equivalence relation, reflexivity, we need to show that the diagonal map $\tilde{U} \to \tilde{U} \times_R \tilde{U}$ factors through $\tilde{Q}$. This follows from the result for schemes. The other axioms are proved similarly.

The quotient algebraic space $\tilde{U}/\tilde{Q}$, see [37, Tag 02WW], is the required extension of X.

The remaining two parts follow from the known result for schemes by lifting the morphisms on atlases. Note that a morphism of algebraic spaces is smooth or étale if the induced morphism on atlases is. □

4.2. **Spreading out algebraic stacks.**

Theorem 4.2. *Let R be an integral domain with function field K. Let $\mathfrak{X}$ be an algebraic stack of finite type over K. Then there is an algebraic stack $\tilde{\mathfrak{X}}$ over an open subscheme $U \subseteq \mathrm{Spec} R$ whose generic fiber is $\mathfrak{X}$.*

Proof. Once again, this is a standard result so we provide a brief sketch of the argument only.

Consider a presentation $X \to \mathfrak{X}$. In view of the hypothesis on $\mathfrak{X}$ the scheme X can be chosen to be of finite type. As smooth morphisms are of finite presentation we have that all the products $X \times_{\mathfrak{X}} \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X$ are of finite type over K. This data forms a groupoid in algebraic spaces $(X, X \times_{\mathfrak{X}} X, \pi_1, \pi_2), \pi_{13})$, [37, Tag 0231], [37, Tag 0437] and [25, 2.4.3]. We can find an open subscheme $U \subseteq \mathrm{Spec} R$ to which all the defining data of this groupoid lifts. We require this lift to form a groupoid, which amounts to certain diagrams commuting as in [25, 2.4.3]. By further refinement, we can find $U' \subseteq U \subseteq \mathrm{Spec} R$ over which we have a groupoid in schemes. To this groupoid in schemes there is an associated algebraic stack, see [25, 4.3.1]. This algebraic stack lifts $\mathfrak{X}$. □

Corollary 4.3. *We work in the situation of the previous theorem. Let $x \in \mathrm{Br}(\mathfrak{X})$ be a Brauer class. Then there is an open subscheme $V \subseteq U$ and a Brauer class $\overline{x} \in \mathrm{Br}(\tilde{\mathfrak{X}}|_V)$ extending x.*

Proof. This follows from 3.1. □

4.3. **The Brauer-Manin pairing on algebraic stacks.** In this subsection we will construct the Brauer-Manin pairing on an algebraic stack of finite type over a number field. Before doing so, let's recall a few facts pertaining to the construction of the pairing for schemes.

For each $v \in \Omega_k$, local class field theory constructs a morphism

$$\mathrm{inv}_{k,v} : \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z},$$

known as the Hasse invariant. The subscripts k and v will frequently be dropped from the notation when the context makes them clear. For finite places, this is an isomorphism. Further there is a short exact sequence

$$0 \to \mathrm{Br}(k) \to \bigoplus_{v \in \Omega_k} \mathrm{Br}(k_v) \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z}.$$

If X is a finite type scheme over k then given an adelic point $x \in X(\mathbf{A}_k)$ we obtain for each $v \in \Omega_k$ a point $x_v \in X(k_v)$ by restricting along the projection $\mathbf{A}_k \to k_v$. The scheme can be spread out to a scheme over $\mathcal{O}_{k,S}$ using 4.1. All but finitely many of the x_v will lie inside $X(\mathcal{O}_v)$ where $v \notin S$.

Given a Brauer class $b \in \mathrm{Br}(X)$, we can also spread it out to a Brauer class over $\mathcal{O}_{k,S}$ by potentially increasing S, see 4.3. As $\mathrm{Br}(\mathcal{O}_v) = 0$ we obtain a pairing

$$X(\mathbf{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$$

given by

$$< x, b >= \sum_{v \in \Omega_k} \mathrm{inv}_{k,v}(x_v^*(b)).$$

The sum is finite in view of the remarks above.

Now let $\mathfrak{X}$ be a finite type algebraic stack over k. We can assume that it lifts to an algebraic stack over $\mathcal{O}_{k,S}$. Let $\mathbf{A}_k$ be the ring of adeles of k. Given an adelic point $x \in \mathfrak{X}(\mathbf{A}_k)$ we obtain for each $v \in \Omega_k$ a k_v-point by restricting along the projection

$$\mathbf{A}_k \to k_v.$$

We will denote this point by x_v. All but finitely many of these will lift to $\mathcal{O}_v$-points.

We define a pairing, the Brauer-Manin pairing,

$$<,>: \mathfrak{X}(\mathbf{A}_k) \times Br(\mathfrak{X}) \to \mathbb{Q}/\mathbb{Z}$$

given by

$$< (x_v), b >= \sum_{v \in S} \mathrm{inv}_{k,v}\, x_v^*(b).$$

Proposition 4.4. *The pairing constructed above exists, in other words the sum is finite.*

Proof. By 4.2 there is a $f \in \mathcal{O}_k$ so that the stack $\mathfrak{X}$ lifts to a stack $\tilde{\mathfrak{X}}$ over $\mathcal{O}_k[f^{-1}]$. We can further assume that the Brauer class lifts to $\tilde{\mathfrak{X}}$ by 4.3. There are only finitely many primes outside $\mathrm{Spec}(\mathcal{O}_k[f^{-1}])$ and the Brauer class $x_v^*(b)$ vanishes for each $v \in \mathrm{Spec}(\mathcal{O}_k[f^{-1}])$. The reason is that the Brauer group $\mathrm{Br}(\mathcal{O}_v) = 0$ is trivial. □

For a subset $B \subseteq \mathrm{Br}(\mathfrak{X})$ we denote by $\mathfrak{X}(\mathbf{A}_k)^B$ the B-fixed point locus (or B-orthogonal locus) of this pairing. It is the subset

$$\mathfrak{X}(\mathbf{A}_k)^B := \{x \in \mathfrak{X}(\mathbf{A}_k) \mid < b, x >= 0,\ \forall\, b \in B\}.$$

4.4. **Continuity of the Brauer-Manin pairing.**

Proposition 4.5. *Let k_v be a local field. Let $\mathfrak{X}$ a finite type liftable algebraic stack over k_v. Let $b \in \mathrm{Br}(\mathfrak{X})$. Then the map*

$$\mathfrak{X}(k_v) \to \mathbb{Q}/\mathbb{Z}$$

given by

$$x \mapsto \mathrm{inv}(x^*(b))$$

is locally constant.

Proof. First assume that $\mathfrak{X}$ is an algebraic space. Using [12, Proposition 5.4] we can find a finite type scheme X and étale morphism $f : X \to \mathfrak{X}$ so that a point $\overline{x} \in \mathfrak{X}(k_v)$ lifts to a point $x \in X$. The morphism $X \to \mathfrak{X}$ is a local homeomorphism. By [34, pg. 235] there is an open neighborhood of x on which the map

$$y \mapsto \mathrm{inv}(y^* f^* b)$$

is constant. The result for algebraic spaces follows.

Now suppose $\mathfrak{X}$ is a finite type liftable algebraic stack. Fix a point $\overline{x} \in \mathfrak{X}(k_v)$. There is a presentation by a finite type algebraic space $f : X \to \mathfrak{X}$ such that $\overline{x}$ lifts to a point $x \in X(k_v)$. By the above, there is an open subset $U \subseteq X(k_v)$ in the v-adic topology on which the map

$$f^*b : X(k_v) \to \mathbb{Q}/\mathbb{Z}$$

is constant. The result follows from the definition of the topology on $\mathfrak{X}(k_v)$, see subsection 2.4. $\square$

Proposition 4.6. *Let $\mathfrak{X}$ be an algebraic stack over a number field k. Assume that $\mathfrak{X}$ is S-liftable and of finite type. For each $b \in \mathrm{Br}(\mathfrak{X})$ the function*

$$\mathfrak{X}(\mathbf{A}_k) \to \mathbb{Q}/\mathbb{Z}$$

given by

$$x \mapsto < b, x >$$

is locally constant.

Hence the fixed point locus

$$\mathfrak{X}(\mathbf{A}_k)^b = \{x \,|< b, x >= 0\}$$

is closed (and open).

Proof. By 4.2, we may assume that $\mathfrak{X}$ is defined over some $\mathcal{O}_{k,T}$. Further we can assume that b is defined over $\mathcal{O}_{k,T}$ by 4.3.

Let $x \in \mathfrak{X}(\mathbf{A}_k)$. By considering projections

$$\mathbf{A}_k \to k_v$$

we obtain k_v-points of $\mathfrak{X}$ written x_v. We can find a finite subset $T \subseteq S \subseteq \Omega_k$ so that $x_v \in \mathfrak{X}(\mathcal{O}_v)$ for $v \notin S$. For each $v \in S$ we can find an open neighbourhood of $x_v \in \mathfrak{X}(k_v)$, denoted U_v, on which evaluation at b is constant. Then evaluation at b is constant on the open subset

$$\prod_{v \in S} U_v \times \prod_{v \notin S} \mathfrak{X}(\mathcal{O}_v) \overset{\text{open}}{\subseteq} \mathfrak{X}(k^S).$$

This is also open in the adelic points of $\mathfrak{X}$, see 2.17. $\square$

4.5. **Approximation on stacks.**

Definition 4.7. Let S be a finite subset of Ω_k and suppose $\mathfrak{X}$ is a finite type algebraic stack over k such that $\mathfrak{X} \otimes_k k_v$ is k_v-liftable for each $v \in S$. We say that *weak approximation for $\mathfrak{X}$ holds at S* if the diagonal image of $\mathfrak{X}(k)$ inside

$$\prod_{v \in S} \mathfrak{X}(k_v)$$

is dense when the above is equipped with the product topology.

The following theorem boils down to the work of M. Borovoi.

Theorem 4.8. *Let S be a finite collection of finite places in Ω_k. Let G be a connected linear algebraic group over k. Then weak approximation holds for BG at S.*

Proof. We fix a faithful representation $G \hookrightarrow \mathrm{GL}_n$. Then we have a liftable presentation

$$f : \mathrm{GL}_n /G \to BG$$

by 2.2. By [6, Corollary 2.5], GL_n is connected, weak approximation holds for GL_n /G at S. If

$$U \subseteq \prod_{v \in S} BG(k_v)$$

is open then $f^{-1}(U)$ is open in $\prod_{v\in S} \mathrm{GL} /G(k_v)$ and hence contains a rational point x, say. Then $f(x)$ is a rational point in U. □

Definition 4.9. Let S be a finite subset of Ω_k and let $\mathfrak{X}$ be a S-liftable algebraic stack over k. If B is a subgroup of $\mathrm{Br}(\mathfrak{X})$ then we say that *strong approximation for $\mathfrak{X}$ holds off S with respect to B* if the diagonal image of $\mathfrak{X}(k)$ is dense inside $\mathfrak{X}(\mathbf{A}_{\not S,k})^B$.

Our results below will verify this property for certain stacks, in particular for classifying stacks of connected linear algebraic groups over k.

5. Strong approximation for quotient stacks

5.1. **Some facts about** $\mathrm{SL}(V)$**.** We let V be a finite dimensional k-vector space and denote by $\mathrm{SL}(V)$ the special linear group of V.

Proposition 5.1. *Let X be a smooth k-variety. Consider the projection map $X \times_k \mathrm{SL}(V) \to X$. Then it induces isomorphisms*

(1) $H^0(X, \mathbb{G}_{\mathrm{m}}) \cong H^0(X \times_k \mathrm{SL}(V), \mathbb{G}_{\mathrm{m}})$,
(2) $H^1(X, \mathbb{G}_{\mathrm{m}}) \cong H^1(X \times_k \mathrm{SL}(V), \mathbb{G}_{\mathrm{m}})$,
(3) $H^2(X, \mathbb{G}_{\mathrm{m}}) \cong H^2(X \times_k \mathrm{SL}(V), \mathbb{G}_{\mathrm{m}})$.

Proof. For a variety, Y over k, the Leray spectral sequence for the morphism $Y \to k$ has

$$E_2^{pq} = H^p(k, H^q(Y \otimes_k \overline{k}, \mathbb{G}_{\mathrm{m}})) \implies H^{p+q}(Y, \mathbb{G}_{\mathrm{m}}).$$

The result will follow by comparison of spectral sequence once we have assembled some results on the cohomology $(X \times \mathrm{SL}(V)) \otimes_k \overline{k}$. Now

$$H^0(X \otimes_k \overline{k}, \mathbb{G}_{\mathrm{m}}) \cong H^0((X \times \mathrm{SL}(V)) \otimes_k \overline{k}, \mathbb{G}_{\mathrm{m}})$$

by [36, 6.5]. Note that in the notation of Sansuc, $U(\mathrm{SL}(V)) := k[\mathrm{SL}(V)]^\times/k^\times = 0$. This follows from [35, theorem 3], using the fact that $\mathrm{SL}(V)$ is equal to it's derived subgroup. Further,

$$H^1(X \otimes_k \overline{k}, \mathbb{G}_{\mathrm{m}}) \cong H^1((X \times \mathrm{SL}(V)) \otimes_k \overline{k}, \mathbb{G}_{\mathrm{m}})$$

by [36, 6.6] and [36, 6.9]. Note that $\mathrm{SL}(V \otimes_k \overline{k})$ is rational. It remains to show that

$$H^2(X \otimes_k \overline{k}, \mathbb{G}_{\mathrm{m}}) \cong H^2((X \times \mathrm{SL}(V)) \otimes_k \overline{k}, \mathbb{G}_{\mathrm{m}}).$$

In view of the proved result for H^1, it suffices to show that

$$H^2(X \otimes_k \overline{k}, \mu_n) \cong H^2((X \times \mathrm{SL}(V)) \otimes_k \overline{k}, \mu_n)$$

using the Kummer sequence

$$1 \to \mu_n \to \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}} \to 1$$

and the fact that the groups in question are torsion. Using the comparison theorem for cohomology, see [2, Exp. XI], we have

$$H^1(\mathrm{SL}(V_{\overline{k}}), \mu_n) = H^2(\mathrm{SL}(V_{\overline{k}}), \mu_n) = 0$$

by [28, pg. 148, Thm 6.5] and the universal coefficient theorems. The equality of H^2 with μ_n-coefficients results from the Kunneth formula. The final result is deduced by comparison of Leray spectral sequences for $X \to k$ and $X \times \mathrm{SL}(V) \to k$. □

Corollary 5.2. *In the above situation, consider the projection map $f : X \times \mathrm{SL}(V) \to X$. Then we have*

(1) $f_*\mathbb{G}_\mathrm{m} \cong \mathbb{G}_\mathrm{m}$,
(2) $R^1 f_*\mathbb{G}_\mathrm{m} = 0$,
(3) $R^2 f_*\mathbb{G}_\mathrm{m} = 0$.

Proof. The etale site has enough points so the statements can be checked on stalks. Let $x : \mathrm{Spec}(\overline{k}) \to X$ be a geometric point. Denote by $\mathcal{O}^{sh}_{X,x}$ the strict Henselisation of the local ring at the image of x. We have a diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}^{sh}_{X,x}) \times \mathrm{SL}(V) & \xrightarrow{p} & X \times \mathrm{SL}(V) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}^{sh}_{X,x}) & \longrightarrow & X. \end{array}$$

We have $p^{-1}\mathbb{G}_\mathrm{m} = \mathbb{G}_\mathrm{m}$, [37, Tag 04DI]. The stalks can be computed as

$$H^i(\mathrm{Spec}(\mathcal{O}^{sh}_{X,x}) \times \mathrm{SL}(V), \mathbb{G}_\mathrm{m}),$$

see [37, Tag 03Q7]. Furthermore, let N_x be the category of affine étale neighbourhood of x. So that

$$\mathrm{Spec}(\mathcal{O}^{sh}_{X,x}) \times \mathrm{SL}(V) = \lim_{U \in N_x} N \times \mathrm{SL}(V),$$

so we may apply [2, Expose V, 5.7-5.8]. In view of the isomorphisms

$$H^i(U \times \mathrm{SL}(V), \mathbb{G}_\mathrm{m}) \cong H^i(U, \mathbb{G}_\mathrm{m})$$

we obtain

$$H^i(\mathrm{Spec}(\mathcal{O}^{sh}_{X,x}) \times \mathrm{SL}(V), \mathbb{G}_\mathrm{m}) \cong H^i(\mathrm{Spec}(\mathcal{O}^{sh}_{X,x}), \mathbb{G}_\mathrm{m}) = 0 \quad i = 1, 2$$

using the fact that $\mathcal{O}^{sh}_{X,x}$ is local and [27, pg. 148,2.13]. The first assertion is proved in a similar way using the definition of the pushforward of sheaf on the étale topos. □

5.2. **Strong Approximation.** We will equip Artin stacks with their big étale site. This has the advantage over the lisse-etale site in that it is functorial. Recall that we have defined the Brauer group of a stack to be the cohomological Brauer group in this topology.

In this section G/k will be a connected linear algebraic group. Fix a faithful representation $G \hookrightarrow \mathrm{SL}(V)$. Such a representation always exists. For example, in view of the fact that G is linear we can find a representation $G \hookrightarrow \mathrm{GL}_n$. Then we may compose it with the representation $\mathrm{GL}_n \hookrightarrow \mathrm{SL}_{2n}$ given by

$$M \longmapsto \begin{pmatrix} M & 0 \\ 0 & (M^t)^{-1} \end{pmatrix}.$$

Given a quotient stack of the form $[Y/G]$ we may write it as an $\mathrm{SL}(V)$-quotient

$$[Y/G] \cong [Y \times_G \mathrm{SL}(V)/\mathrm{SL}(V)],$$

see 2.1. The object $X := Y \times_G \mathrm{SL}(V)$ always exists as an algebraic space. If G is reductive and Y is quasi-projective with a linearised action then the quotient will be a scheme, see [30].

Proposition 5.3. *Let X be a smooth variety with an action of $\mathrm{SL}(V)$. Consider the morphism*

$$f : X \to [X/\mathrm{SL}(V)].$$

Then

$$f^* : \mathrm{Br}([X/\mathrm{SL}(V)]) \to \mathrm{Br}(X)$$

is an isomorphism.

Proof. We have a cartesian diagram

$$\begin{tikzcd} X \times \mathrm{SL}(V) \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & {[X/\mathrm{SL}(V)]}\,. \end{tikzcd}$$

The top horizontal map is the action and the left vertical map is projection. There is an automorphism of $X \times \mathrm{SL}(V)$ that switches these, so both maps are smooth. We may apply the spectral sequence [37, Tag 06XJ] to both of these maps. The spectral sequence for $X \to [X/\mathrm{SL}(V)]$ has

$$E_1^{pq} = H^q(X \times \mathrm{SL}(V)^{\times p}, \mathbb{G}_{\mathrm{m}}).$$

The spectral sequence for $X \times \mathrm{SL}(V) \to X$ has

$$E_1^{pq} = H^q(X \times \mathrm{SL}(V)^{\times p+1}, \mathbb{G}_{\mathrm{m}}).$$

The induced maps

$$H^q(X \times \mathrm{SL}(V)^{\times q}, \mathbb{G}_{\mathrm{m}}) \to H^q(X \times \mathrm{SL}(V)^{\times q+1}, \mathbb{G}_{\mathrm{m}}).$$

are induced by projections so the result follows from 5.1. $\square$

Using 2.2 every adelic point of $[Y \times_G \mathrm{SL}(V)/\mathrm{SL}(V)]$ lifts to $Y \times_G \mathrm{SL}(V)$. In other words, the presentation

$$Y \times_G \mathrm{SL}(V) \to [Y \times_G \mathrm{SL}(V)/\mathrm{SL}(V)]$$

is S-liftable for various S.

Recall from earlier in this section, given a linear algebraic group G we can always find a faithful representation

$$G \hookrightarrow \mathrm{SL}(V).$$

Theorem 5.4. *Let G be a linear algebraic group. Fix a faithful representation*

$$G \hookrightarrow \mathrm{SL}(V).$$

Let S be a finite subset of Ω_k. Suppose $Y \times_G \mathrm{SL}(V)$ is smooth and that strong approximation holds for $Y \times_G \mathrm{SL}(V)$ holds off S with respect to $\mathrm{Br}(Y \times_G \mathrm{SL}(V))$. Then strong approximation holds for $[Y \times_G \mathrm{SL}(V)/\mathrm{SL}(V)]$ off S with respect to $\mathrm{Br}([Y \times_G \mathrm{SL}(V)/\mathrm{SL}(V)])$.

Proof. In what follows we will write f for the presentation

$$Y \times_G \mathrm{SL}(V) \to [Y \times_G \mathrm{SL}(V)/\mathrm{SL}(V)].$$

We will abuse notation and write f for the map induced on R-points for various rings R.

Let $x \in [Y \times_G \mathrm{SL}(V)/\mathrm{SL}(V)](\mathbf{A}_{\not S,k})$ be a point orthogonal to $\mathrm{Br}([Y \times_G \mathrm{SL}(V)/\mathrm{SL}(V)])$. We can lift it to a point $y \in Y \times_G \mathrm{SL}(V)(\mathbf{A}_{\not S,k})$. In view of the above corollary, if x is orthogonal to $\mathrm{Br}([Y \times_G \mathrm{SL}(V)/\mathrm{SL}(V)])$ then y is orthogonal to $\mathrm{Br}(Y \times_G \mathrm{SL}(V))$ as the Brauer-Manin pairing is easily seen to be functorial, i.e

$$< x, b >=< y, f_* b > .$$

If U is an open neighbourhood of x in the adelic topology then $f^{-1}(U)$ is an open neighbourhood of y in the adelic topology. Then there a k-point $t \in Y \times_G \mathrm{SL}(V)(k)$ in $f^{-1}(U)$. It follows that $f(t)$ is a k-point in U. □

To generate examples to which this theorem applies, we may use the ideas of [8]. A variety with X with G-action is said to be groupic if there is an dense equivariant open subset of X that is isomorphic to G as a G-variety. It is easy to see that if X is G-groupic then $X \times_G \mathrm{SL}(V)$ is $\mathrm{SL}(V)$ -groupic as

$$G \times_G \mathrm{SL}(V) \cong \mathrm{SL}(V).$$

The main theorem of [8, 1.3] can be used to generate examples. In this vein, quotients of toric varieties by tori can be use to provide examples.

We change direction now and consider classifying stacks, which are not covered by the above examples. This result is in contrast to the above result for quotient stacks as strong approximation, as we have defined it, does not hold for homogeneous spaces in general. In [7, 6.1] and [11, 3.7(b)] a kind of equivariant strong approximation theorem for homogeneous spaces is proved which is exactly what is needed in order to prove strong approximation for classifying spaces.

Theorem 5.5. *Let G be a connected linear algebraic group. Let S be a finite set of places of k that contains at least one finite place and all archimedean places. Then strong approximation for BG off S with respect to* $\mathrm{Br}(BG)$ *holds.*

Proof. Fix a faithful representation

$$G \hookrightarrow \mathrm{SL}(V).$$

Let $X = G \setminus \mathrm{SL}(V)$ which is smooth. Then there is a presentation

$$f : X \to [X/\mathrm{SL}(V)] \cong BG.$$

The isomorphism is from 2.1. We will abuse notation and write f for the map induced on R-points for various rings R.

Now consider a point $\bar{x} \in BG(\mathbf{A}_S)^{\mathrm{Br}(BG)}$ which can be lifted to a point $x \in X(\mathbf{A}_S)^{\mathrm{Br}(X)}$, using 5.3 and 2.2.

Let U be an adelic neighbourhood of $\bar{x}$ so that $f^{-1}(U)$ is a neighbourhood of x. Observe that strong approximation holds for $\mathrm{SL}(V)$, see [24] and [32]. So we may apply [7, Theorem 6.1]. By this theorem we can find a rational point $t \in X(k)$ and a point $g \in G(k_S)$ so that $g.t$ belongs to $f^{-1}(U)$. Now, $f(t) = f(gt) \in U$. Further, although gt is not k-rational, we have that $f(gt) = f(t) \in U$ is. Hence the result. □

5.3. **A counterexample for disconnected groups.** The aim of this subsection is to show that the connectedness assumption is essential in 5.5. We let $\xi \in \mathbb{C}$ be a primitive cube root of unity. In this subsection $k = \mathbb{Q}(\xi)$. Our specific goal is the following theorem.

Theorem 5.6. *Let $k = \mathbb{Q}(\xi)$. Let S be the set of archimedean (i.e infinite) places of k. Then strong approximation fails for $B\mu_2$ with respect to $\mathrm{Br}(B\mu_2)$ off S.*

The proof will be a lengthy computation. The calculation will show that the failure of strong approximation holds more generally, that is, one could prove similar results for other μ_n. We leave these generalisations to the reader. We do note that strong approximation with respect to the Brauer group is known for $\mathbb{G}_\mathrm{m}$, see [9], which should be seen as motivation for the computations that follow. We prove and recall some preliminary results before presenting the proof of the theorem at the end of the subsection.

There is a faithful representation $\mu_2 \hookrightarrow \mathbb{G}_\mathrm{m}$ and the quotient is $\mathbb{G}_\mathrm{m} = \mathbb{G}_\mathrm{m}/\mu_2$. It follows that $B\mu_2 = [\mathbb{G}_\mathrm{m}/\mathbb{G}_\mathrm{m}]$ where the action is via the square map

$$\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m} \qquad z \mapsto z^2.$$

We write $f : \mathbb{G}_\mathrm{m} \to B\mu_2$ for the corresponding liftable presentation.

Lemma 5.7. *In the above situation, let R be a ring over which every $\mathbb{G}_\mathrm{m}$-torsor is trivial, for example the adele ring (see 2.2) or a local ring. Then every R-point $\overline{p} \in B\mu_2(R)$ lifts to $\mathbb{G}_\mathrm{m}$. Further two R-points of $\mathbb{G}_\mathrm{m}$ say p_1 and $p_2 \in R^\times$ satisfy*

$$f(p_1) = f(p_2)$$

if and only if there is a $\lambda \in R^\times$ with $\lambda^2 p_1 = p_2$.

Proof. The proof is straightforward. □

To proceed we need to describe the image of the map

$$f^* : \mathrm{Br}(B\mu_2) \to \mathrm{Br}(\mathbb{G}_\mathrm{m}).$$

The Brauer group of $B\mu_2$ has been described in [1, §3] and also [26, §4]. We recall their results and refer to the articles for proofs. There is a split short exact sequence

$$0 \to \mathrm{Br}(k) \to \mathrm{Br}(B\mu_2) \to H^1(k, \mu_2) \to 0.$$

One can describe the splitting in the following way. We have the usual Kummer sequence for H^1:

$$0 \to \{\pm 1\} \to k^\times \xrightarrow{(-)^2} k^\times \to H^1(k, \mu_2) \to 1.$$

Given a non-square $d \in k$ then consider the matrix

$$\begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix}.$$

It has order two in PGL_2 so that it defines an action of μ_2 on the matrix algebra $M_{2\times 2}$. The corresponding Azumaya algebra on $B\mu_2$ is the splitting. We denote it by $A(d)$. Technically, we have defined the Brauer group of an algebraic stack to be it's cohomological Brauer group, however, the Azumaya algebra definition agrees with cohomological definition in this case, see [26].

Next we recall the description of the Brauer group of $\mathbb{G}_\mathrm{m} = \mathrm{Spec}(k[t, t^{-1}])$. The general description can be found in [18]. However, we really only a description of the 2-torsion and 3-torsion parts of the Brauer group. This can be made more explicit by using cyclic algebras. The 2-torsion is generated by the 2-torsion in $\mathrm{Br}(k)$ and the cyclic algebras

$$(d,t)_{-1} \quad \text{where} \quad d \in k^{\times}$$

and t is the parameter on $\mathbb{G}_{\mathrm{m}}$. Likewise the 3-torsion is generated 3-torsion in $\mathrm{Br}(k)$ and the cyclic algebras

$$(d,t)_{\xi} \quad \text{where} \quad d \in k^{\times},$$

see [20, 2.7]. We claim that $f^*A(d) = (d,t)_{-1}$ so that the pullback map on 2-torsion is surjective. The reason is that the algebra $f^*A(d)$ is the quotient of the trivial algebra on $\mathbb{G}_{\mathrm{m}}$ by the μ_2-action defined by the matrix described above. But this is exactly a cyclic algebra, see [19, 2.5]. Technically the calculation is for fields but this makes no difference by [18, 3.1].

Further, modulo $\mathrm{Br}(k)$, the image of f^* is entirely 2-torsion.

We write $\mathrm{inv}(\alpha,\beta)_{\omega} \in \mathbb{Q}/\mathbb{Z}$ for the Hasse invariant of a cyclic algebra over a local or global field. We will make frequent use of the fact that $(\alpha,\beta\gamma)_{\omega} = (\alpha,\beta)_{\omega} + (\alpha,\gamma)_{\omega}$ in the Brauer group, see [16, Lemma 4, pg. 72].

We will need the following technical lemma.

Lemma 5.8. *Recall that $k = \mathbb{Q}(\xi)$ where ξ is a primitive cube root of unity. There is a finite place $v \in \Omega_k$, an integer $n \in \mathcal{O}_k$ and an element $x \in k_v^{\times}$ so that*

(1) $\mathrm{inv}(n,x)_{\xi} = \frac{2}{3}$
(2) $\mathrm{inv}(d,x)_{-1} = 0$ *for all* $d \in k$.

Proof. Let $n \in \mathcal{O}_k$ be a non-cube. Then consider L the splitting field of $X^3 - n$. There are only finitely many primes of $\mathcal{O}_k$ that ramify in L, so let v be an umramified prime. If π is a local parameter for $\mathcal{O}_v$ then consider the cyclic algebra $A = (n,\pi)_{\xi}$. It satisfies

$$\mathrm{inv}\, A = \frac{1}{3}.$$

We take $x = \pi^2$. Then $(n,x)_{\xi}$ has the same Brauer class as A^2 so the first property follows. Further the element $(d,\pi)_{-1}$ is 2 -torsion for all $d \in k$. Hence the second property follows. □

Proof. (of 5.6)

We have a previously described lifting presentation

$$f : \mathbb{G}_{\mathrm{m}} \to B\mu_2.$$

We let $n,\ v$ and x be as in the previous lemma. Let U be the subset of $\mathbb{G}_{\mathrm{m}}(\mathbf{A}_{k,S})$ consisting of $(x_w) \in \mathbb{G}_{\mathrm{m}}(\mathbf{A}_{k,S})$ so that

$$\mathrm{inv}(x_w)^*(n,t)_{\xi} = 2/3.$$

By [34, 8.2.11] the set U is open. Next we show that U is non-empty. Indeed consider the idele defined by

$$x_w = \begin{cases} 1 & w \neq v \\ x & w = v. \end{cases}$$

As $\mathrm{Br}(\mathcal{O}_w) = 0$, it follows by the previous lemma that $(x_w) \in U$. Let B be the 2-torsion subgroup of $\mathrm{Br}(\mathbb{G}_\mathrm{m})$. Recall that it is generated by cyclic algebras of the form $(d,t)_{-1}$ with $d \in k^\times$. By the lemma, we have

$$\mathbb{G}_\mathrm{m}(\mathbf{A}_{k,S})^B \cap U \neq \emptyset. \tag{①}$$

We claim that

$$y^2 U \cap \mathbb{G}_\mathrm{m}(k) = \emptyset \quad \forall y \in \mathbb{G}_\mathrm{m}(\mathbf{A}_{k,S}). \tag{②}$$

To see this take $x \in U$ then

$$\mathrm{inv}((n, y^2x)_\xi) = 2\,\mathrm{inv}((n,y)_\xi) + \mathrm{inv}((n,x)_\xi) \neq 0$$

as $\mathrm{inv}((n,y)_\xi)$ is 3-torsion. The result follows from the fact that elements of $\mathbb{G}_\mathrm{m}(k)$ are orthogonal to $\mathrm{Br}(\mathbb{G}_\mathrm{m})$.

Consider the open set $f(U)$ inside $B\mu_2(\mathbf{A}_{k,S})$. As the pullback map

$$f^* : \mathrm{Br}(B\mu_2) \to \mathrm{Br}(\mathbb{G}_\mathrm{m})$$

is surjective on 2-torsion and $\mathrm{Br}(B\mu_2)$ is generated by the image of $\mathrm{Br}(k)$ and the 2-torsion subgroup, we see that

$$f(U) \cap B\mu_2(\mathbf{A}_{k,S})^{\mathrm{Br}(B\mu_2)} \neq \emptyset$$

by ①.

On the other hand, it is non-empty and contains no rational points. To see this, if $\overline{p}$ is a k-rational point of $B\mu_2$ then it lifts to a k-rational point of $\mathbb{G}_\mathrm{m}$ by 5.7. Further two points p and x of $\mathbb{G}_\mathrm{m}(\mathbf{A}_{k,S})$ satisfy $f(p) = f(x)$ if and only if there is a $y \in \mathbb{G}_\mathrm{m}(\mathbf{A}_{k,S})$ so that $y^2 p = x$, by 5.7. But such an x can never be in U by ②.

It follows that $f(U)$ has no k-rational points. □

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