Fields Lectures:
Presheaves of spectra

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May 7, 2007

This is a preliminary version of these notes. Corrected versions are sure to follow.

Contents

*This research was supported by NSERC.

1 Spectra

A *spectrum* X consists of pointed (level) simplicial sets X^n, $n \geq 0$ together with *bonding maps* $\sigma : S^1 \wedge X^n \to X^{n+1}$.

A*map* $f : X \to Y$ *of spectra* consists of pointed maps $f : X^n \to Y^n$ which respect structure in the sense that the diagrams

$$\begin{array}{ccc} S^1 \wedge X^n & \xrightarrow{\sigma} & X^{n+1} \\ {\scriptstyle S^1 \wedge f}\downarrow & & \downarrow{\scriptstyle f} \\ S^1 \wedge Y^n & \xrightarrow[\sigma]{} & Y^{n+1} \end{array}$$

commute. The corresponding category will be denoted by **Spt**. This category is complete and cocomplete.

Examples:

- Suppose that Y is a pointed simplicial set. The *suspension spectrum* $\Sigma^\infty Y$ consists of the pointed sets

  $$Y,\ S^1 \wedge Y,\ S^1 \wedge S^1 \wedge Y,\ \dots\ , S^n \wedge Y, \dots$$

  where $S^n = S^1 \wedge \cdots \wedge S^1$ (n-fold smash power). The bonding maps of $\Sigma^\infty Y$ are the canonical isomorphisms

  $$S^1 \wedge S^n \wedge Y \cong S^{n+1} \wedge Y.$$

  There is a natural bijection

  $$\hom(\Sigma^\infty Y, X) \cong \hom(X, Y^0),$$

  so that the suspension spectrum functor is left adjoint to the "level 0" functor $X \mapsto X^0$.

- $S = \Sigma^\infty S^0$ is the *sphere spectrum*.

- $X =$ spectrum and $K =$ pointed simplicial set: there is a spectrum $X \wedge K$ with

  $$(X \wedge K)^n = X^n \wedge K$$

  and having bonding maps

  $$\sigma \wedge K : S^1 \wedge X^n \wedge K \to X^{n+1} \wedge K.$$

  $\Sigma^\infty K \cong S \wedge K$. The *suspension* of a spectrum X is the spectrum $X \wedge S^1$.

- The *fake suspension* ΣX of X has level spaces $S^1 \wedge X^n$ and bonding maps

  $$S^1 \wedge \sigma : S^1 \wedge S^1 \wedge X^n \to S^1 \wedge X^{n+1}.$$

- The *fake loop spectrum* ΩY has level spaces ΩY^n and (adjoint) bonding maps
$$\Omega\sigma_* : \Omega Y^n \to \Omega^2 Y^{n+1}.$$
The fake loops functor $Y \mapsto \Omega Y$ is right adjoint to the fake suspension functor $X \mapsto \Sigma X$.

- Suppose that X is a spectrum and K is a pointed simplicial set. There is a spectrum $\mathbf{hom}_*(K, X)$ with
$$\mathbf{hom}_*(K, X)^n = \mathbf{hom}_*(K, X^n),$$
and with bonding map
$$S^1 \wedge \mathbf{hom}_*(K, X^n) \to \mathbf{hom}_*(K, X^{n+1})$$
adjoint to the composite
$$S^1 \wedge \mathbf{hom}_*(K, X^n) \wedge K \xrightarrow{S^1 \wedge ev} S^1 \wedge X^n \xrightarrow{\sigma} X^{n+1}.$$
There is a natural bijection
$$\hom(X \wedge K, Y) \cong \hom(X, \mathbf{hom}_*(K, Y)).$$

- Suppose that X is a pointed simplicial set. Write $\tilde{\mathbb{Z}}(X)$ for the kernel of the map $\mathbb{Z}(X) \to \mathbb{Z}(*)$. Then $H_n(X, \mathbb{Z}) = \pi_n(\mathbb{Z}(X), 0)$ and $\tilde{H}_n(X, \mathbb{Z}) = \pi_n(\tilde{\mathbb{Z}}(X), 0)$ (reduced homology). There is a natural pointed map
$$X \xrightarrow{\eta} \mathbb{Z}(X) \to \tilde{\mathbb{Z}}(X),$$
denoted by h for "Hurewicz". If A is a simplicial abelian group, there is a natural simplicial set map
$$\gamma : S^1 \wedge A \to \tilde{\mathbb{Z}}(S^1) \otimes A =: S^1 \otimes A.$$
The *Eilenberg-Mac Lane* spectrum $H(A)$ associated to A consists of the spaces
$$A,\ S^1 \otimes A,\ S^2 \otimes A,\ \dots$$
with bonding maps
$$S^1 \wedge (S^n \otimes A) \xrightarrow{\gamma} S^1 \otimes (S^n \otimes A) \cong S^{n+1} \otimes A.$$

Suppose that X is a spectrum and $n \in \mathbb{Z}$. The *shifted spectrum* $X[n]$ is the spectrum with
$$X[n]^m = \begin{cases} * & m+n < 0 \\ X^{m+n} & m+n \geq 0 \end{cases}$$

Remarks: 1) There is a natural map

$$\sigma : \Sigma X \to X[1]$$

defined by the bonding maps, and it's easy to see that this map is a stable equivalence. It's more interesting, but still classical (the argument involves a telescope construction) to show that there is a stable equivalence $\Sigma X \simeq X \wedge S^1$. Thus, all flavours of suspension coincide with shift up to stable equivalence.

The adjoint map

$$\sigma_* : X \to \Omega X[1]$$

is a stable equivalence for a spectrum X consisting of Kan complexes X^n.

2) There is a natural bijection

$$\hom(X[n], Y) \cong \hom(X, Y[-n])$$

and of course $X[n][-n] \cong X$, so that all shift operators are invertible.

3) There is a natural bijection

$$\hom(\Sigma^\infty K[-n], Y) \cong \hom(K, Y^n),$$

so that the n^{th} level functor $Y \mapsto Y^n$ has a left adjoint.

4) Given a spectrum X, the n^{th} *layer* L_nX is the spectrum

$$X^0,\ \dots\ , X^n,\ S^1 \wedge X^n,\ S^2 \wedge X^n,\ \dots$$

There are obvious maps $L_nX \to L_{n+1}X \to X$ and a natural isomorphism

$$\varinjlim_n\ L_nX \cong X.$$

The functor $X \mapsto L_nX$ is left adjoint to truncation up to level n. The system of maps

$$\Sigma^\infty X^0 = L_0X \to L_1X \to \dots$$

is called the *layer filtration* of X. Here's an exercise: show that there are pushout diagrams

$$\begin{array}{ccc} \Sigma^\infty(S^1 \wedge X^n)[-n-1] & \longrightarrow & L_nX \\ {\scriptstyle \sigma_*}\downarrow & & \downarrow \\ \Sigma^\infty X^{n+1}[-n-1] & \longrightarrow & L_{n+1}X \end{array}$$

2 Presheaves of spectra

A *presheaf of spectra* X is a functor

$$X : \mathcal{C}^{op} \to \mathbf{Spt}.$$

Alternatively, X consists of pointed simplicial presheaves X^n, $n \geq 0$ together with bonding maps

$$\sigma : S^1 \wedge X^n \to X^{n+1},\ n \geq 0.$$

Here, S^1 is identified with the constant pointed simplicial presheaf $U \mapsto S^1$.

A map $f : X \to Y$ of presheaves of spectra consists of pointed simplicial presheaf maps $f : X^n \to Y^n$, $n \geq 0$, which respect structure in the obvious sense. Write $\mathrm{Spt}(\mathcal{C})$ for the category of presheaves of spectra on $\mathcal{C}$.

Examples:

1) If I is a small category, then the category of I-diagrams $X : I \to \mathbf{Spt}$ is a category of presheaves of spectra on I^{op}, where I^{op} has the trivial topology. In particular, the ordinary category **Spt** of spectra is the category of presheaves of spectra on the one-object, one-morphism category.

2) A spectrum Y (of pointed simplicial sets) determines an associated constant presheaf of spectra $\Gamma^* Y$ on $\mathcal{C}$, where

$$\Gamma^* Y(U) = Y,$$

and every morphism $\phi : V \to U$ induces the identity morphism $Y \to Y$. We shall often write $Y = \Gamma^* Y$ when there is no possibility of confusion. The sphere spectrum S in $\mathrm{Spt}(\mathcal{C})$ is the constant object $\Gamma^* S$ associated to the ordinary sphere spectrum.

The functor $Y \mapsto \Gamma^* Y$ is left adjoint to the global sections functor $\Gamma_* : \mathrm{Spt}(\mathcal{C}) \to \mathbf{Spt}$, where

$$\Gamma_* X = \varprojlim_{U \in \mathcal{C}} X(U).$$

3) If A is a sheaf (or presheaf) of abelian groups, the Eilenberg-Mac Lane presheaf of spectra $H(A)$ is the presheaf of spectra underlying the suspension object

$$A, S^1 \otimes A, S^2 \otimes A, \dots$$

in the category of presheaves of spectra in simplicial abelian groups. Note that , as a simplicial presheaf, $S^n \otimes A = K(A, n)$.

If $j : K(A,n) \to FK(A,n)$ is a globally fibrant model of $K(A,n)$ then there are natural isomorphisms

$$\pi_j \Gamma_* FK(A,n) = \begin{cases} H^{n-j}(\mathcal{C}, A) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

We'll see later that these isomorphisms assemble to give an identification of the stable homotopy groups of global sections of a (stably) fibrant model for $H(A)$ with the cohomology of $\mathcal{C}$ with coefficients in A.

Slogan: All sheaf cohomology groups are stable homotopy groups.

4) More generally, every chain complex (bounded or unbounded) D determines a presheaf of spectra $H(D)$, which computes the hypercohomology of $\mathcal{C}$ with coefficients in D, via computing stable homotopy groups of global sections of a

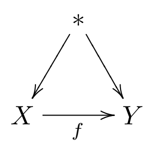
fibrant model. Spectrum objects in presheaves of simplicial R-modules give a model for the full derived category.

5) There is a presheaf of spectra K on $\mathrm{Sch}|_S$, called the algebraic K-theory spectrum, such that $\pi_j K(U)$ is the j^{th} algebraic K-group $K_j(U)$. The construction of this object is still rather unsatisfactory, after all these years: in its most general form, it starts with a pseudo-functor on S-schemes taking values in pseudo-simplicial symmetric monoidal categories [6], [7]. I will give a simpler version of this construction later in these lectures.

3 Model structures

For much of what follows $\mathcal{C}$ will be an arbitrary small Grothendieck site.

Write $s_* \operatorname{Pre}(\mathcal{C})$ for the category of pointed simplicial presheaves $* \to X$, with base point preserving maps



Recall that this category has a proper closed simplicial model structure for which a map f as above is a *local weak equivalence* (resp. *global fibration*, *cofibration*) if and only if the underlying map $f : X \to Y$ of simplicial presheaves is a local weak equivalence (resp. global fibration, cofibration). The function complex $\mathbf{hom}(X, Y)$ is defined in simplicial degree n by

$$\mathbf{hom}(X, Y)_n = \text{all maps } X \wedge \Delta^n_+ \to Y$$

(Z_+ always means Z union a disjoint base point). In particular a cofibration of simplicial presheaves is a map which is a monomorphism in each simplicial degree and each section. In the presence of a meaningful theory of stalks, a map $f : X \to Y$ is a local weak equivalence if and only if it induces weak equivalences $f : X_x \to Y_x$ of simplicial sets in all stalks.

Warning: A map f of pointed simplicial presheaves is a local weak equivalence if and only if it induces an isomorphism on sheaves of path components, and induces isomorphisms

$$\tilde{\pi}_n(X|_U, x) \to \tilde{\pi}_n(Y|_U, f(x))$$

for all $U \in \mathcal{C}$ and all (local) choices of base points $x \in X(U)$. It is **not** sufficient to check that the obvious maps $\tilde{\pi}_n(X, *) \to \tilde{\pi}_n(Y, *)$ (based at the canonical point $* \to X$) are isomorphisms.

Say that a map $f : X \to Y$ of presheaves of spectra is a *strict weak equivalence* (respectively *strict fibration*) if all maps $f : X^n \to Y^n$ are local equivalences (respectively global fibrations).

A *cofibration* $i : A \to B$ of $\operatorname{Spt}(\mathcal{C})$ is a map for which

- $i : A^0 \to B^0$ is a cofibration, and
- all maps
$$(S^1 \wedge B^n) \cup_{(S^1 \wedge A^n)} A^{n+1} \to B^{n+1}$$
are cofibrations.

One can show that $i : A \to B$ is a cofibration of spectra if and only if all maps $A^n \to B^n$ and all maps $S^1 \wedge (B^n/A^n) \to B^{n+1}/A^{n+1}$ are cofibrations of pointed simplicial sets.

The function complex $\mathbf{hom}(X, Y)$ for presheaves of spectra X, Y is defined in simplicial degree n in the usual way:

$$\mathbf{hom}(X, Y)_n = \text{all maps } X \wedge \Delta^n_+ \to Y.$$

Lemma 3.1. *With these definitions, the category $s\,\mathrm{Pre}(\mathcal{C})$ satisfies the axioms for a proper closed simplicial model category.*

The proof is an exercise. Of course, Lemma 3.1 is just an opening act.

A presheaf of spectra X has presheaves $\pi_n X$ of stable homotopy groups, defined by
$$U \mapsto \pi_n X(U).$$
Write $\tilde{\pi}_n X$ for the sheaf associated to the presheaf $\pi_n X$. The sheaves $\tilde{\pi}_n X$, $n \in \mathbb{Z}$, are the sheaves of stable homotopy groups of X.

Say that a map $f : X \to Y$ of presheaves of spectra is a *stable equivalence* if it induces isomorphisms
$$\tilde{\pi}_n X \xrightarrow{\cong} \tilde{\pi}_n Y$$
for all $n \in \mathbb{Z}$.

Observe that every strict equivalence is a stable equivalence.

Say that $p : Z \to W$ is a *stable fibration* if it has the right lifting property with respect to all maps which are cofibrations and stable equivalences.

Remark 3.2. These definitions of cofibration, stable equivalence and stable fibration are direct generalizations of (and specialize to) corresponding definitions given by Bousfield and Friedlander [1] for the category of spectra. In particular, a map $i : A \to B$ is a cofibration of presheaves of spectra if and only if all maps $A(U) \to B(U)$ are cofibrations of spectra.

Theorem 3.3. *With the definitions of cofibration, stable equivalence and stable cofibration given above, the category $\mathrm{Spt}(\mathcal{C})$ satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*

Lemma 3.4. *A map $p : X \to Y$ is a stable fibration and a stable equivalence if and only if all maps $p : X^n \to Y^n$ are trivial global fibrations of simplicial presheaves.*

Proof. One implication is a consequence of the strict model structure.

The other is a factorization argument. Suppose that p is a stable fibration and a stable equivalence. Then p has a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & W \\ & \searrow^{p} & \downarrow q \\ & & Y \end{array}$$

where j is a cofibration and q is a trivial strict fibration. But then j is stable equivalence as well as a cofibration, so that the lifting exists in the diagram

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ W & \xrightarrow[q]{} & Y \end{array}$$

so that p is a retract of q and is therefore a trivial strict fibration. □

Suppose that α is a cardinal number. Say that a presheaf of spectra A is *α-bounded* if all pointed sets $A^n_m(U)$, $m, n \geq 0$, $U \in \mathcal{C}$ satisfy $|A^n_m(U)| < \alpha$. Observe that every presheaf of spectra X is a union of its α-bounded subobjects.

Now suppose that α is an infinite cardinal such that $|\operatorname{Mor}(\mathcal{C})| < \alpha$.

We have to know something basic about cofibrations:

Lemma 3.5. *Suppose that $i : X \to Y$ is a cofibration of spectra, and that A is a subobject of Y. Then the induced map $A \cap X \to A$ is a cofibration of spectra.*

Proof. If $A \subset Y$, certainly all maps $A^n \cap X^n \to A^n$ are cofibrations, and there are commutative diagrams

$$\begin{array}{ccc} S^1 \wedge (A^n/(A^n \cap X^n)) & \longrightarrow & A^{n+1}/(A^{n+1} \cap X^{n+1}) \\ \downarrow & & \downarrow \\ S^1 \wedge Y^n/X^n & \longrightarrow & Y^{n+1}/X^{n+1} \end{array}$$

in which the vertical maps are cofibrations. □

The next result asserts that the stably trivial cofibrations satisfy a "bounded cofibration condition":

Lemma 3.6. *Suppose given a cofibration $i : X \to Y$ which is a stable equivalence, and suppose that A is an α-bounded subobject of Y. Then there is an α-bounded subobject B of Y such that $A \subset B$ and the cofibration $B \cap X \to B$ is a stable equivalence.*

Proof. Note that $\tilde{\pi}_n Z = 0$ if and only if for all $x \in \pi_n Z(U)$ there is a covering sieve $\phi : V \to U$ such that $\phi^*(x) = 0$ for all ϕ in the covering.

The sheaves $\tilde{\pi}_n(Y/X)$ are trivial (sheafify the natural long exact sequence for a cofibration), and

$$\tilde{\pi}_n(Y/X) = \varinjlim_{C} \tilde{\pi}_n(C/C \cap X)$$

where C varies over all α-bounded subobjects of Y. The list of elements of all $x \in \pi_n(A/A \cap X)(U)$ is α-bounded. For each such x there is an α-bounded subobject $B_x \subset X$ such that

$$x \mapsto 0 \in \tilde{\pi}_n(B_x/B_x \cap X).$$

It follows that there is an α-bounded subobject

$$B_1 = A \cup (\cup_x B_x)$$

such that all $x \mapsto 0 \in \tilde{\pi}_n(B_1/B_1 \cap X)$.

Write $A = B_0$. Then inductively, we can produce an ascending sequence

$$A = B_0 \subset B_1 \subset B_2 \subset \dots$$

of α-bounded subobjects of Y such that all presheaf homomorphisms

$$\pi_n(B_i/B_i \cap X) \to \tilde{\pi}_n(B_{i+1}/B_{i+1} \cap X)$$

are trivial. Set $B = \cup_i B_i$. Then B is α-bounded and all sheaves $\tilde{\pi}_n(B/B \cap X)$ are trivial. □

Lemma 3.7. *The class of stably trivial cofibrations has a generating set, namely the set I of all α-bounded stably trivial cofibrations.*

The proof of this result amounts to the verification of a "solution set condition", with the category theory language excised.

Proof. By Lemma 3.4, the class of cofibrations of $\mathrm{Spt}(\mathcal{C})$ is generated by the set J of cofibrations

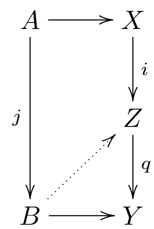
$$\Sigma^\infty A[-n] \to \Sigma^\infty B[-n]$$

which are induced by α-bounded cofibrations $A \to B$ of pointed simplicial presheaves.

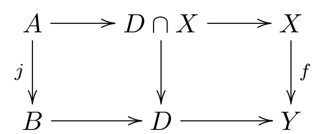
Suppose given a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

where j is a cofibration, B is α-bounded, and f is a stable equivalence. Then f has a factorization $f = q \cdot i$ where i is a cofibration and q is a strictly trivial fibration, hence a stable equivalence, and the lifting exists in the diagram



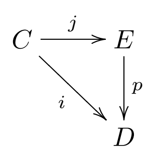
The cofibration $i : X \to Z$ is a stable equivalence, and the image $\theta(B) \subset Z$ is α-bounded, so there is an α-bounded subobject $D \subset Z$ with $\theta(B) \subset D$ such that $D \cap X \to D$ is a stable equivalence. It follows that there is a factorization



of the original diagram through an α-bounded stably trivial cofibration.

In particular, if f is a stable equivalence which has the right lifting property with respect to all α-bounded stably trivial cofibrations, then f has the right lifting property with respect to all cofibrations.

Now suppose that $i : C \to D$ is a stably trivial cofibration. Then i has a factorization



where j is a cofibration in the saturation of the set of α-bounded stably trivial cofibrations and p has the right lifting property with respect to all α-bounded stably trivial cofibrations. The map j is a stable equivalence since the class of stably trivial cofibrations is closed under pushout (by a long exact sequence argument) and composition. Then p is a stable equivalence and has the right lifting property with respect to all α-bounded cofibrations; p therefore has the right lifting property with respect to all cofibrations. It follows that i is a retract of the map j. □

Proof of Theorem 3.3. The model structure and the cofibrant generation follow from Lemmas 3.4 – 3.7. The argument for the simplicial model axiom **SM7** is standard: one shows by induction on n that if $i : A \to B$ is a stably trivial cofibration then all maps

$$i \wedge \partial\Delta^n_+ : A \wedge \partial\Delta^n_+ \to B \wedge \partial\Delta^n_+$$

are stable equivalences. Left and right properness are consequences of long exact sequences in stable homotopy groups. □

Since the stable model structure on $\mathrm{Spt}(\mathcal{C})$ is cofibrantly generated there is a functorial stably fibrant model construction.

$$j : X \to LX$$

Note that if X and Y are stably fibrant, any stable equivalence $f : X \to Y$ must be a strict equivalence — this is a consequence of Lemma 3.4. It follows that a map $f : X \to Y$ of arbitrary presheaves of spectra is a stable equivalence if and only if the induced map $LX \to LY$ is a strict equivalence, and so a map $f : X \to Y$ is a stable equivalence if and only if it is an "L-equivalence".

We also have the following:

A4 The functor L preserves strict equivalences.

A5 The maps $j_{LX}, Lj_X : LX \to LLX$ are strict weak equivalences.

A6′ Stable equivalences are preserved by pullback along stable fibrations.

Here is a recognition principle for stable fibrations:

Theorem 3.8. *A map $p : X \to Y$ of $\mathrm{Spt}(\mathcal{C})$ is a stable fibration if and only it is a strict fibration and the diagram*

$$\begin{array}{ccc} X & \xrightarrow{j} & LX \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Lp} \\ Y & \xrightarrow[j]{} & LY \end{array}$$

is strictly homotopy cartesian.

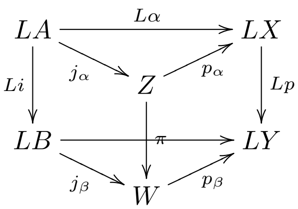
The following result encapsulates one of the basic ideas in the proof of Theorem 3.8. The other main ingredient is the properness of the stable model structure.

Lemma 3.9. *Suppose that $p : X \to Y$ is a strict fibration, and that the maps $j : X \to LX$ and $j : Y \to LY$ are strict equivalences. Then p is a stable fibration.*

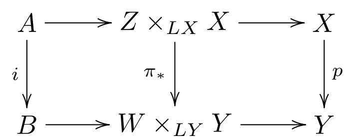
Proof. Consider the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\beta]{} & Y \end{array}$$

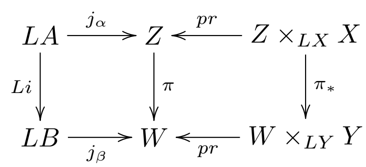
There is a diagram



where j_α, j_β are strictly trivial cofibrations and p_α, p_β are strict fibrations. There is an induced diagram



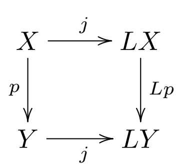
and the lifting problem is solved if we can show that π_* is a strict weak equivalence. But there is finally a diagram



The maps Li, j_α and j_β are strict weak equivalences so that π is a strict weak equivalence. The maps pr are strict weak equivalences by right properness and the assumptions on X and Y. It follows that π_* is a strict weak equivalence. □

Corollary 3.10. *Suppose that $p : X \to Y$ is a strict fibration between stably fibrant objects. Then p is a stable fibration.*

Proof of Theorem 3.8. Suppose that p is a strict fibration and that the diagram



is strictly homotopy cartesian. The map Lp has a factorization $Lp = q \cdot i$ where $q : Z \to LY$ is a stable fibration and $i : LX \to Z$ is a cofibration and a stable equivalence. Then i is a strict equivalence since LX and Z are stably fibrant. The pullback $q_* : Y \times_{LY} Z \to Y$ is a stable fibration, and the induced map $\theta : X \to Y \times_{LY} Z$ is a strict equivalence since the square is strictly homotopy cartesian. Then p is a stable fibration by Lemma 3.9.

Suppose that p is a stable fibration, and factorize the map Lp as $Lp = q \cdot i$ as above, ie. so that $q : Z \to LY$ is a stable fibration and i is a stably trivial cofibration. Then the induced map $\theta : X \to Y \times_{LY} Z$ is a stable equivalence *since j pulls back to a stable equivalence along* q, and in fact determines a stable equivalence between stably fibrant objects of $\mathrm{Spt}(\mathcal{C})/Y$. The map θ is therefore a strict equivalence. □

Lemma 3.11. *Suppose that $p : X \to Y$ is a stable fibration. Then the diagrams*

$$\begin{CD} X^n @>{\sigma_*}>> \Omega X^{n+1} \\ @V{p}VV @VV{\Omega p}V \\ Y^n @>>{\sigma_*}> \Omega Y^{n+1} \end{CD}$$

are strictly homotopy cartesian.

Proof. Since p is a stable fibration, any stably trivial cofibration $\theta : A \to B$ induces a homotopy cartesian diagram

$$\begin{CD} \mathbf{hom}(B,X) @>{p_*}>> \mathbf{hom}(B,Y) \\ @V{\theta^*}VV @VV{\theta^*}V \\ \mathbf{hom}(A,X) @>>{p_*}> \mathbf{hom}(A,Y) \end{CD}$$

If $\theta : A \to B$ is a stable equivalence between cofibrant objects, then the diagram above is still homotopy cartesian. In effect, θ has a factorization $\theta = \pi \cdot j$ where j is a trivial cofibration and $\pi \cdot i = 1$ for some trivial cofibration i. It follows that the diagram above is a retract of a homotopy cartesian diagram, and is therefore homotopy cartesian.

The diagrams of the statement arise from the stable equivalences $\Sigma^\infty S^1[-1-n] \to S[-n]$. □

Corollary 3.12. *If X is stably fibrant, then all simplicial presheaves X^n are globally fibrant and all adjoint bonding maps $\sigma_* : X^n \to \Omega X^{n+1}$ are sectionwise weak equivalences.*

Note that a map $f : X \to Y$ of globally fibrant simplicial presheaves is a local weak equivalence if and only if it is a sectionwise weak equivalence.

In effect, every trivial global fibration is a sectionwise trivial fibration, and every map $f : X \to Y$ between globally fibrant simplicial presheaves can be factored $f = p \cdot j$ where p is a global fibration and j is a section of a trivial global fibration. This follows from a standard construction: form the pullback

$$\begin{CD} X \times_Y Y^I @>{pr}>> Y^I @>{d_1}>> Y \\ @V{d_{0*}}VV @VV{d_0}V \\ X @>>{f}> Y \end{CD}$$

where $Y^I = \mathbf{hom}(\Delta^1, Y)$. Then $p = d_1 \cdot pr$ and $j : X \to X \times_Y Y^I$ is induced by the constant homotopy $s_0 : Y \to Y^I$, and is therefore a section of the trivial fibration d_{0*}.

Proposition 3.13. *A presheaf of spectra X is stably fibrant if and only if all level objects X^n are globally fibrant and all adjoint bonding maps $\sigma_* : X^n \to \Omega X^{n+1}$ are local weak equivalences.*

Proof. Suppose that all X^n are globally fibrant and all $\sigma_* : X^n \to \Omega X^{n+1}$ are local weak equivalences. Then the simplicial presheaves X^n and ΩX^{n+1} are globally fibrant and cofibrant, so that all σ_* are homotopy equivalences. It follows that all spaces $X^n(U)$ are fibrant and that all maps $\sigma_* : X^n(U) \to \Omega X^{n+1}(U)$ are weak equivalences of pointed simplicial sets, and hence that all maps

$$\pi_k X^n(U) \to \pi_{k-n} X(U)$$

taking values in stable homotopy groups are isomorphisms.

Suppose that $j : X \to LX$ is a stably fibrant model for X. Then all spaces $LX^n(U)$ are fibrant and all maps $LX^n(U) \to \Omega LX^{n+1}(U)$ are weak equivalences, and so all maps

$$\pi_k LX^n(U) \to \pi_{k-n} LX(U)$$

are isomorphisms. The map j induces an isomorphism in all sheaves of stable homotopy groups, and so the maps $j : X^n \to LX^n$ induce isomorphisms

$$\tilde{\pi}_k X^n \to \tilde{\pi}_k LX^n$$

of sheaves of homotopy groups for $k \geq 0$. The spectrum objects X and LX are presheaves of infinite loop spaces, and so the maps $X^n \to LX^n$ are local weak equivalences of simplicial presheaves. In particular, $j : X \to LX$ is a strict weak equivalence.

The result then follows from Lemma 3.9. □

Remark 3.14. In other words, a stably fibrant presheaf of spectra X is a presheaf of Ω-spectra such that each level object X^n is globally fibrant.

Write $\Omega^\infty X$ for the filtered colimit of the system of maps

$$X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega\sigma_*} \Omega^2 X[2] \xrightarrow{\Omega^2\sigma_*} \dots$$

in the category $\mathrm{Spt}(\mathcal{C})$. Write $j : Y \to FY$ for a natural strictly fibrant model.

Corollary 3.15. *The presheaf of spectra*

$$QX = F\Omega^\infty FX$$

is stably fibrant, for any presheaf of spectra X.

Proof. The natural map $\eta : X \to QX$ defined by the composite

$$X \to FX \to \Omega^\infty FX \to F\Omega^\infty FX$$

is a stable equivalence. The object $F\Omega^\infty FX$ is stably fibrant, since the loop functor $Y \mapsto \Omega Y$ preserves local weak equivalences between presheaves of pointed Kan complexes. □

4 Fibrations and cofibrations

Suppose that $i : A \to X$ is a levelwise cofibration of spectra with cofibre $\pi : X \to X/A$. Suppose that $\alpha : S^r \to X^n$ represents a homotopy element such that the composite

$$S^r \xrightarrow{\alpha} X^n \xrightarrow{\pi} X^n/A^n$$

represents $0 \in \pi_r(X/A)^n$. Then, by comparing cofibre sequences, there is a commutative diagram

$$\begin{array}{ccccccc}
S^r & \longrightarrow & CS^r & \longrightarrow & S^1 \wedge S^r & \xrightarrow{\simeq} & S^1 \wedge S^r \\
{\scriptstyle \alpha}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle S^1\wedge\alpha} \\
X^n & \xrightarrow[\pi]{} & (X/A)^n & \longrightarrow & S^1 \wedge A^n & \xrightarrow{S^1\wedge i} & S^1 \wedge X^n \\
 & & & & {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\
 & & & & A^{n+1} & \xrightarrow[i]{} & X^{n+1}
\end{array}$$

where $CS^r \simeq *$ is the cone on S^r. It follows that the image of $[\alpha]$ under the suspension map

$$\pi_r X^n \to \pi_{r+1} X^{n+1}$$

is in the image of the map $\pi_{r+1}A^{n+1} \to \pi_{r+1}X^{n+1}$. We have proved the following:

Lemma 4.1. *Suppose that $A \to X \to X/A$ is a levelwise cofibre sequence of spectra. Then the sequence*

$$\pi_k A \to \pi_k X \to \pi_k(X/A)$$

is exact.

Corollary 4.2. *Any levelwise cofibre sequence*

$$A \to X \to X/A$$

induces a long exact sequence

$$\dots \xrightarrow{\partial} \pi_k A \to \pi_k X \to \pi_k(X/A) \xrightarrow{\partial} \pi_{k-1} A \to \dots$$

Proof. The map $X/A \to A \wedge S^1$ in the Puppe sequence induces the boundary map

$$\pi_k(X/A) \to \pi_k(A \wedge S^1) \cong \pi_k(A[1]) \cong \pi_{k-1}A.$$

since $A \wedge S^1$ is naturally stably equivalent to the shifted spectrum $A[1]$. □

The long exact sequence for a cofibre sequence of spectra is natural, and hence determines a long exact sequence

$$\dots \xrightarrow{\partial} \tilde{\pi}_k A \to \tilde{\pi}_k X \to \tilde{\pi}_k(X/A) \xrightarrow{\partial} \tilde{\pi}_{k-1}A \to \dots$$

in sheaves of stable homotopy groups for a cofibre sequence $A \to X \to X/A$ of presheaves of spectra. This long exact sequence is natural in cofibre sequences of presheaves of spectra.

There is also a natural long exact sequence in stable homotopy groups

$$\dots \xrightarrow{\partial} \pi_k F \to \pi_k E \to \pi_k B \xrightarrow{\partial} \pi_{k-1}F \to \dots$$

for a levelwise fibre sequence of spectra

$$F \to E \to B.$$

This follows (by taking appropriate filtered colimits) for the comparisons of fibre sequences

$$\begin{array}{ccccccccc} \dots & \longrightarrow & E^n & \longrightarrow & B^n & & & & \\ & & \downarrow & & \downarrow & & & & \\ \dots & \to & \Omega E^{n+1} & \to & \Omega B^{n+1} & \to & F^{n+1} & \to & E^{n+1} \to B^{n+1} \end{array}$$

Again a naturality argument implies that there is a long exact sequence

$$\dots \xrightarrow{\partial} \tilde{\pi}_k F \to \tilde{\pi}_k E \to \tilde{\pi}_k B \xrightarrow{\partial} \tilde{\pi}_{k-1}F \to \dots$$

in sheaves of stable homotopy groups for a levelwise fibre sequence of presheaves of spectra

$$F \to E \to B.$$

This long exact sequence is natural in levelwise fibre sequences of presheaves of spectra.

Corollary 4.3. *Suppose that X and Y are presheaves of spectra. Then the inclusion $X \vee Y \to X \times Y$ is a natural stable equivalence.*

Proof. The sequence

$$0 \to \tilde{\pi}_k X \to \tilde{\pi}_k(X \vee Y) \to \tilde{\pi}_k Y \to 0$$

arising from the level cofibration $X \subset X \vee Y$ is split exact, as is the sequence

$$0 \to \tilde{\pi}_k X \to \tilde{\pi}_k(X \times Y) \to \tilde{\pi}_k Y \to 0$$

arising from the fibre sequence $X \to X \times Y \to Y$. It follows that the map $X \vee Y \to X \times Y$ induces an isomorphism in all sheaves of stable homotopy groups. □

Corollary 4.4. *The stable homotopy category* $\mathrm{Ho}(\mathrm{Spt}(\mathcal{C}))$ *is additive: the sum of two maps* $f, g : X \to Y$ *is represented by the composite*

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y \xleftarrow{\simeq} Y \vee Y \xrightarrow{\nabla} Y.$$

Example 4.5. The map $\times n : X \to X$ which is multiplication by n in the stable category is defined in the stable category by the composite

$$X \xrightarrow{\Delta} \prod_n X \xleftarrow{\simeq} \bigvee_n X \to \nabla X.$$

and the presheaf of spectra X/n is the homotopy cofibre of this map, so that there is a cofibre sequence

$$X \xrightarrow{\times n} X \to X/n.$$

Evaluating in sections gives cofibre sequences of ordinary spectra

$$X(U) \xrightarrow{\times n} X(U) \to X/n(U),$$

so that $X/n(U) \simeq X(U)/n$. The multiplication by n map $\times n : X \to X$ induces multiplication by n in stable homotopy groups (hence in sheaves of stable homotopy groups), so the cofibre sequence induces short exact sequences

$$0 \to \tilde{\pi}_k(X) \otimes \mathbb{Z}/n \to \tilde{\pi}_k X \to {}_n\tilde{\pi}_{k-1}(X) \to 0,$$

where the thing on the right denotes the n-torsion in $\tilde{\pi}_{k-1}(X)$. In the world of ordinary spectra, one often writes

$$\pi_k(X, \mathbb{Z}/n) = \pi_k(X/n).$$

- The mod n K-groups $K_i(S, \mathbb{Z}/n)$ of a scheme S are the stable homotopy groups of the spectrum $K(S)/n$ constructed from the algebraic K-theory spectrum $K(S)$ of S.
- The spectrum S/n obtained from the sphere spectrum S has a special name: it's the mod n *Moore spectrum.*

Slogan: Fibre and cofibre sequences coincide in the stable category:

Lemma 4.6. *Suppose that* $A \xrightarrow{i} X \xrightarrow{\pi} X/A$ *is a pointwise cofibre sequence in* $\mathrm{Spt}(\mathcal{C})$*, and let* F *be the strict homotopy fibre of the map* $\pi : X \to X/A$*. Then the induced map* $i_* : A \to F$ *is a stable equivalence.*

Proof. Choose a strict fibration $p : Z \to X/A$ such that $Z \to *$ is a strict weak equivalence. Form the pullback

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\pi_*} & Z \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[\pi]{} & X/A \end{array}$$

Then $\tilde{X}$ is the homotopy fibre of π and the maps $i : A \to X$ and $* : A \to Z$ together determine a map $i_* : A \to \tilde{X}$. We show that i_* is a stable equivalence.

Pull back the cofibre square

$$\begin{array}{ccc} A & \longrightarrow & * \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \xrightarrow[\pi]{} & X/A \end{array}$$

along the fibration p to find a (levelwise) cofibre square

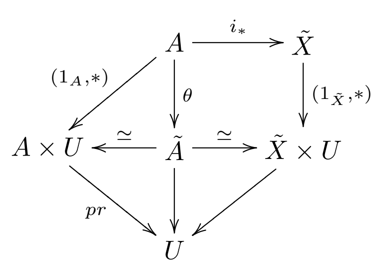
$$\begin{array}{ccc} \tilde{A} & \longrightarrow & U \\ {\scriptstyle \tilde{i}}\downarrow & & \downarrow \\ \tilde{X} & \xrightarrow[\pi_*]{} & Z \end{array}$$

The spectrum Z is contractible, so a Mayer-Vietoris sequence argument implies that the map $\tilde{A} \to \tilde{X} \times U$ is a stable equivalence.

Also, from the fibre square

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & U \\ \downarrow & & \downarrow \\ A & \longrightarrow & * \end{array}$$

we see that the map $\tilde{A} \to A \times U$ is a stable equivalence. The map $i_* : A \to \tilde{X}$ induces a section $\theta : A \to \tilde{A}$ of the map $\tilde{A} \to A$ which composes with the projection $\tilde{A} \to U$ to give the trivial map $* : A \to U$. It follows that there is a commutative diagram



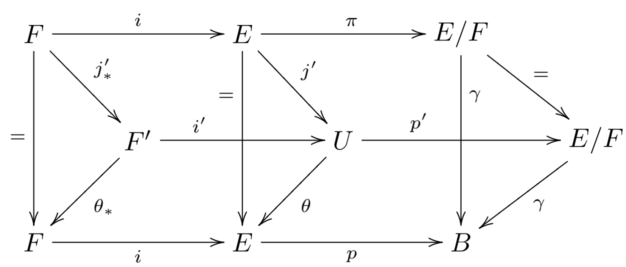
It follows that A is the stable homotopy fibre of the map $\tilde{A} \to U$, and so i_* is a stable equivalence. □

Lemma 4.7. *Suppose that*

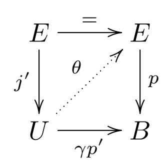
$$F \xrightarrow{i} E \xrightarrow{p} B$$

is a strict fibre sequence, where i *is a levelwise cofibration. Then the induced map* $\gamma : E/F \to B$ *is a stable equivalence.*

Proof. There is a diagram



where p' is a strict fibration, j' is a cofibration and a strict equivalence, and θ exists by a lifting property:



Then the map j'_* is a stable equivalence by Lemma 4.6, so that θ_* is a stable equivalence. The map θ is a strict equivalence, so it follows from a comparison of long exact sequences in stable homotopy groups that γ is a stable equivalence. □

5 Descent

A *stably fibrant model* of a presheaf of spectra F is a stably trivial cofibration $j : F \to Z$ such that Z is stably fibrant.

I say that a presheaf of spectra F *satisfies descent* if some (hence any) stably fibrant model $j : F \to Z$ induces stable equivalences $F(U) \to Z(U)$ in all sections (this is a *sectionwise* stable equivalence).

Why should we care about descent? Why should we care about stably fibrant objects?

Suppose that Z is stably fibrant and that X is a pointed simplicial presheaf. Then there is a spectrum $\mathbf{hom}(X, Z)$ with spaces at level n given by the pointed function complexes $\mathbf{hom}(X, Z^n)$ and with bonding maps

$$\mathbf{hom}(X, Z^n) \xrightarrow{\sigma_*} \mathbf{hom}(X, \Omega Z^{n+1}) \cong \Omega\mathbf{hom}(X, Z^{n+1}).$$

The maps $Z^n \to \Omega Z^{n+1}$ are local weak equivalences of globally fibrant objects, and therefore induce weak equivalences

$$\mathbf{hom}(X, Z^n) \to \mathbf{hom}(X, \Omega Z^{n+1}).$$

Also, all pointed simplicial sets $\mathbf{hom}(X, Z^n)$ are Kan complexes. All of this follows from the fact that pointed simplicial presheaves has a closed simplicial model structure. In particular, the spectrum $\mathbf{hom}(X, Z)$ is stably fibrant — it's an Ω-spectrum.

But the same general nonsense says that any local weak equivalence $X \to Y$ of pointed simplicial presheaves induces a stable (even levelwise) weak equivalence

$$\mathbf{hom}(Y, Z) \to \mathbf{hom}(X, Z).$$

Example: Suppose that S is some scheme and that $T \to S$ is an S-scheme which is locally of finite type; as such it is an object of the category $Sch|_S$ of S-schemes which are locally of finite type. Write $(Sch|_S)_{et}$ for the "big" site of S-schemes with the étale topology.

Let T represent a constant simplicial presheaf on the big étale site. A *hypercover* $V \to T$ is most generally defined to be simplicial presheaf map which is a trivial Kan fibration in all stalks. It is, in any case a weak equivalence. Examples include Čech resolutions $C(U) \to T$ associated to étale covers $U \to T$ of T.

A hypercover is, in any case, a local weak equivalence. Thus, if Z is a stably fibrant presheaf of spectra on $(Sch|_S)_{et}$, any hypercover $V \to T$ induces a stable equivalence

$$Z(T) \cong \mathbf{hom}(T_+, Z) \to \mathbf{hom}(V_+, Z).$$

If V has the good manners to be representable, and this can be arranged up to refinement, then $\mathbf{hom}(V, Z)$ is a realization (or total object) of the cosimplicial spectrum $Z(V_n)$, and so by the usual technology of cosimplicial spaces there is a spectral sequence with

$$E_2^{s,t} = H^s \pi_t Z(V)$$

which "converges" to $\pi_{t-s} F(T)$. This is the "finite" descent spectral sequence for $Z(T)$ associated to the hypercover $V \to T$.

The most well known particular case of this general story arises when L/k is a finite Galois extension with Galois group G, and $S = \mathrm{Sp}(k)$. In that case $\mathrm{Sp}(L) \to Sp(k)$ is an étale covering,

$$L \otimes_k L \cong \prod_{g \in G} L$$

by Galois theory and the Čech resolution for the covering $\mathrm{Sp}(L) \to \mathrm{Sp}(k)$ can be identified up to isomorphism with the Borel construction $EG \times_G \mathrm{Sp}(L)$. In that case the finite descent spectral sequence for the stable homotopy groups $\pi_* Z(k)$ takes the form

$$E_2^{s,t} = H^s(G, \pi_t Z(L))$$

and "converges" to $\pi_{t-s} Z(k)$. One often says that the weak equivalence

$$Z(k) \simeq \mathbf{hom}(EG \times_G \mathrm{Sp}(L), Z) = \underleftarrow{\mathrm{holim}}_G Z(L)$$

identifies $Z(k)$ with the homotopy fixed points for the action of G on $Z(L)$.

The finite descent spectral sequence is not "the" descent spectral sequence, although there's been some wishful thinking about that over the years.

In the most general setting, recall that if A is an abelian sheaf and if $K(A,n) \to FK(A,n)$ is a globally fibrant model of the Eilenberg-Mac Lane object $K(A,n)$, then there is a natural isomorphism

$$\pi_j \Gamma_* FK(A,n) = \begin{cases} H^{n-j}(\mathcal{C}, A) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

Here, Γ_* is the global sections functor, aka. inverse limit. To see this, one can use a cocycle argument to show that

$$\pi_0 \Gamma_* FK(A,n) \cong [*, K(A,n)] \cong H^n(\mathcal{C}, A),$$

and then one picks up the other homotopy groups through the identifications

$$FK(A, n-1) \simeq \Omega FK(A,n).$$

It follows that if $H(A) \to LH(A)$ is a stably fibrant model of the Eilenberg-Mac Lane spectrum $H(A)$, then there is a natural isomorphism

$$\pi_t \Gamma_* LH(A) \cong \begin{cases} H^{-t}(\mathcal{C}, A) & t \leq 0 \\ 0 & t > 0. \end{cases}$$

To go further, we need the Postnikov tower construction for a presheaf of spectra F. In general outline, the Postnikov tower for F consists of maps $F \to P_n F$, $n \in \mathbb{Z}$ together with compatible maps $P_n F \to P_{n-1} F$, such that $\tilde{\pi}_s P_n F = 0$ for $s > n$ and the maps $F \to P_n F$ induce isomorphisms $\tilde{\pi}_s F \to \tilde{\pi}_s P_n F$ for $s \leq n$. The homotopy fibre of the map $P_n F \to P_{n-1} F$ can therefore be identified up to (local) stable equivalence with the shifted Eilenberg-Mac Lane spectrum $H(\tilde{\pi}_n F)[-n]$.

The construction of the spectra $P_n E$ for a spectrum E (which construction is natural) rests on the existence of the string of maps

$$\begin{aligned} S^1 \wedge P_n S|E^m| \to & \Sigma P_n S|E^m| \xrightarrow{\Psi_*} P_{n+1} \Sigma S|E^m| \\ & \to P_{n+1} S|E^{m+1}|. \end{aligned}$$

Here, ΣY is the Kan suspension of Y, which is defined by collapsing of Y inside a cone which is built by gluing together simplices Δ^{n+1}, one for each simplex $\Delta^n \to Y$ of Y. The Kan suspension and smashing with S^1 are related in canonical ways, and in particular there is a natural isomorphism $|\Sigma Y| \cong |S^1 \wedge Y|$.

Every spectrum determines a Kan spectrum (suitably defined) through the singular functor and realization. Finally the combinatorial cone construction respects the Postnikov decomposition for a simplicial set, as displayed by the map Ψ_*.

The overall point is that the natural Postnikov tower is defined for Kan spectra, and the construction is transported to ordinary spectra through standard equivalences. See [7] for details.

Now suppose that E is a presheaf of spectra which is *connective* in the sense that $\tilde{\pi}_s F = 0$ for $s < 0$. Form the Postnikov tower

$$\cdots \to P_2 E \to P_1 E \to P_0 E,$$

and then replace the tower by a fibrant tower, meaning that one inductively forms a picture

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_2 E & \longrightarrow & P_1 E & \longrightarrow & P_0 E \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\ \cdots & \longrightarrow & LP_2 E & \xrightarrow[p]{} & LP_1 E & \xrightarrow[p]{} & LP_0 E \end{array}$$

such that each map j is a stably fibrant model and each p is a stable fibration. Note that the fibre of the map

$$LP_n E \to LP_{n-1} E$$

is a fibrant model for the shifted Eilenberg-Mac Lane spectrum $H(\tilde{\pi}_n E)[n]$ by properness. Then applying global sections gives an induced tower of stable fibrations of spectra

$$\ldots \Gamma_* LP_2 E \to \Gamma_* LP_1 E \to \Gamma_* LP_0 E$$

with fibres given by the objects $\Gamma_* LH(\tilde{\pi}_n F)[n]$. Then the Bousfield-Kan technology for this tower of fibrations (with a suitable reindexing) gives a spectral sequence with

$$E_2^{s,t} = H^s(\mathcal{C}, \tilde{\pi}_t E)$$

"converging" to

$$\pi_{t-s} \varprojlim_n \Gamma_* LP_n E.$$

There are two problems:

- convergence, and
- the map $E \to \varprojlim_n LP_n E$ might not be a weak equivalence, so that the inverse limit, even though it is stably fibrant, might not be a stably fibrant model of E.

Both of these problems are solved in practical situations [18] by the assumption of a uniform bound on cohomological dimension, such as one sees for the étale topology for schemes of finite type over something decent.

Examples:

1) Suppose that ℓ is a prime bigger than 3, and that k is a field which contains a primitive ℓ^{th} root of unity. Suppose that k_{sep}/k has a finite ("Tate-Tsen") filtration

$$k = L_0 \subset L_1 \subset \cdots \subset L_r = k_{sep}$$

by Galois subextensions such that $cd_\ell(L_{i+1}/L_i) \leq 1$. Suppose finally that $cd_l(k) \leq d$. Let $j : K/\ell \to LK/\ell$ be a stably fibrant model for the mod ℓ K-theory presheaf of spectra on the étale site $(Sch|_k)_{et}$ of the field k.

Then under these assumptions the Lichtenbaum-Quillen Conjecture asserts that the stable homotopy groups $\pi_s F$ of the fibre $F(k)$ of the global sections map $j : K/\ell(k) \to LK/\ell(k)$ satisfy

$$\pi_s F(k) = 0 \text{ for } s \geq d-2.$$

Equivalently the map $\pi_s K/\ell(k) \to \pi_s FK/\ell(k)$ is an isomorphism for $i \geq d-1$ and is monic if $i = d-2$.

In particular, the K-theory with torsion coefficients of the field k is computed with Galois cohomology in sufficiently high degrees.

2) Suppose that ℓ is a prime bigger than 3, and that X is a scheme such that all residue fields $k(x)$ satisfy the assumptions of Example 1, and that there is a uniform bound d on $cd_\ell(k(x)$ for all $x \in X$. Let $j : K/\ell \to LK/\ell$ be a stably fibrant model for K/ℓ on $(Sch|_k)_{et}$. If the Lichtenbaum-Quillen conjecture holds for fields, then the map

$$j : \pi_s K/\ell(X) \to \pi_s LK/\ell(X)$$

is an isomorphism for $s \leq d-1$ and is a monomorphism for $s \leq d-2$, where $cd_\ell(X) \leq d$.

In effect, let $j : K/\ell \to LK/\ell$ be a stably fibrant model for K/ℓ on $(Sch|_k)_{et}$. Then the restriction to $et|_X$ is a stably fibrant model for K/ℓ on the étale site for X (restriction is exact, and has a left adjoint which preserves cofibrations and local weak equivalences). Now change topologies to the Nisnevich site $Nis|_X$ by direct image. Then LK/ℓ is stably fibrant for the Nisnevich topology (direct image always preserves stable fibrations), while K/ℓ satisfies Nisnevich descent (Nisnevich descent theorem), so the homotopy fibre F of the map $K/\ell \to LK/\ell$ also satisfies Nisnevich descent. The presheaves of spectra K/ℓ and LK/ℓ also both satisfy Gabber rigidity; for K/ℓ, this means that there are isomorphisms

$$\pi_s K/\ell(\mathcal{O}_x^h) \xrightarrow{\cong} \pi_s K/\ell(k(x))$$

for all points $x \in X$. There is a corresponding isomorphism for $\pi_* LK/\ell$, essentially because the étale sheaves $\tilde{\pi}_* K/\ell$ are constant (see [7]), again by Gabber rigidity. But then $\pi_s F(\mathcal{O}_x^h) = 0$ for $t \geq d-2$ so that $\tilde{\pi}_t F = 0$ for $t \geq d-2$ (Nisnevich topology), whence $\pi_t F(X) = 0$ for $t \geq d-2$ since F satisfies Nisnevich descent.

This argument (essentially due to Thomason) was the earliest serious application of Nisnevich descent.

6 T-spectra and localization

I want to show you how to do several things at once.

Suppose that T is a pointed simplicial presheaf on a small site $\mathcal{C}$.

A *T-spectrum* X is a collection of pointed simplicial presheaves X^n, $n \geq 0$, with pointed maps $\sigma : T \wedge X^n \to X^{n+1}$. A map $f : X \to Y$ of T-spectra consists of pointed simplicial presheaf maps $f : X^n \to Y^n$ which respect structure in the obvious way. Write $\mathrm{Spt}_T(\mathcal{C})$ for the category of T-spectra.

Example 6.1.

- The *sphere spectrum* S_T for the category of T-spectra consists of the pointed simplicial presheaves

$$S^0, T, T^{\wedge 2}, T^{\wedge 3}, \dots$$

with the associativity isomorphisms $T \wedge T^{\wedge n} \cong T^{\wedge n+1}$ as bonding maps.

- Given a pointed simplicial presheaf K, $\Sigma_T^\infty K$ is the T-spectrum

$$K, T \wedge K, T^2 \wedge K, \dots$$

The functor $K \mapsto \Sigma_T^\infty K$ is left adjoint to the 0-level functor $X \mapsto X^0$.

- Given a T-spectrum X, $n \in \mathbb{Z}$, the shifted T-spectrum $X[n]$ is defined by

$$X[n]^k = \begin{cases} X^{n+k} & n+k \geq 0 \\ * & n+k < 0 \end{cases}$$

- If K is a pointed simplicial presheaf and X is a T-spectrum, then $X \wedge K$ has the obvious meaning:

$$(X \wedge K)^n = X^n \wedge K.$$

The function complex $\mathbf{hom}(X, Y)$ for T-spectra X and Y is the simplicial set with

$$\mathbf{hom}(X, Y)_n = \text{all maps } X \wedge \Delta^n_+ \to Y.$$

Say that a map $f : X \to Y$ of T-spectra is a *strict weak equivalence* (resp. *strict fibration*) if all maps $f : X^n \to Y^n$ are local weak equivalences (resp. global fibrations) of pointed simplicial presheaves on $\mathcal{C}$.

A *cofibration* of T-spectra is a map $i : A \to B$ such that

- $i : A^0 \to B^0$ is a cofibration of simplicial presheaves, and
- all maps

$$(T \wedge B^n) \cup_{(T \wedge A^n)} A^{n+1} \to B^{n+1}$$

are cofibrations of simplicial presheaves.

Note that $i : A \to B$ is a cofibration if and only if all maps $A^n \to B^n$ and all maps $T \wedge (B^n / A^n) \to B^{n+1}/A^{n+1}$ are cofibrations.

Example 6.2. The sphere spectrum S_T is cofibrant, as are all of its shifts (cofibrations are preserved by shifts).

There is a distinguished map

$$\omega : \Sigma_T^\infty T[-1] \to S_T$$

which is a levelwise cofibration but is not a cofibration: it is easily seen that the cofibre of this map consists of a copy of S^0 concentrated in level 0, and this spectrum is not cofibrant.

This map ω is also the canonical example of a stable equivalence.

Lemma 6.3. *Suppose given the diagram*

$$\begin{array}{ccc} A \cap X & \longrightarrow & X \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle j} \\ A & \xrightarrow[i]{} & Y \end{array}$$

in spectra, where j is a cofibration and i is a levelwise cofibration. Then the induced map $j_* : A \cap X \to A$ is a cofibration.

Proof. The proof is the same as for Lemma 3.5. □

Lemma 6.4. *With these definitions, the category of $\mathrm{Spt}_T(\mathcal{C})$ of T-spectra on $\mathcal{C}$ satisfies the definitions for a proper closed simplicial model category.*

The proof is the usual thing.

There are various names out there for this structure: this is a strict model structure in the sense of Bousfield and Friedlander [1], while Hovey [3] calls it a projective structure.

Basic Assumptions: Suppose that S is a set of cofibrations such that

- A is cofibrant for all $i : A \to B$ in S.
- S includes the set I of generating maps

  $$\Sigma_T^\infty C[-n] \to \Sigma_T^\infty D[-n],\ n \geq 0,$$

  for the strict trivial cofibrations of $\mathrm{Spt}_T(\mathcal{C})$, which are induced by the α-bounded trivial cofibrations $C \to D$ of pointed simplicial presheaves.
- If $i : A \to B$ is a member of S, then all cofibrations

  $$(A \wedge D) \cup (B \wedge C) \to B \wedge D$$

  induced by i and all α-bounded cofibrations $C \to D$ of pointed simplicial presheaves are in S.

Recall that α is a cardinal such that $|\operatorname{Mor}(\mathcal{C})| < \alpha$, and that the α-bounded cofibrations generate the cofibrations for pointed simplicial presheaves.

A map $p : Z \to W$ is said to be *injective* (or S-*injective*) if it has the right lifting property with respect to all maps of S. An object Z is injective if the map $X \to *$ is injective.

Note that every injective object is strictly fibrant.

Because S is a set, we can make a functorial injective model $j : X \to LX$ by a transfinite small object construction: solve the lifting problem sufficiently many times and you get LX.

Say that a map $f : X \to Y$ of $\operatorname{Spt}(\mathcal{C})$ is an L-*equivalence* if it induces a bijection

$$f^* : [Y, Z] \xrightarrow{\cong} [X, Z]$$

in morphisms in the strict homotopy category for every injective object Z.

Examples:

• Every strict equivalence $X \to Y$ is an L-equivalence.

• A map $f : Z \to W$ between injective objects is an L-equivalence if and only if it is a strict equivalence. In effect, f is an isomorphism in the strict homotopy category, and hence a strict equivalence.

• All cofibrations appearing in the set S are L-equivalences, because we've rigged our definition of S so that any such map $A \to B$ induces a trivial fibration

$$\mathbf{hom}(B, Z) \to \mathbf{hom}(A, Z)$$

for all injective Z.

What's more interesting is the following:

Lemma 6.5. *All cofibrations in the saturation of the set S are L-equivalences.*

The proof boils down to showing that the inductive method for constructing the saturation preserves L-equivalences; one uses cofibrant replacements to see this.

Corollary 6.6. *1) The natural map $j : X \to LX$ is an L-equivalence.*

2) A map $f : X \to Y$ is an L-equivalence if and only if the induced map $Lf : LX \to LY$ is a strict equivalence.

Say that a cofibration is L-*trivial* if it is an L-equivalence.

Lemma 6.7. *There is a cardinal κ for which the set of κ-bounded L-trivial cofibrations is a generating set for the class of L-trivial cofibrations.*

Specifically take $\kappa \geq 2^\lambda$, where $\lambda \geq 2^\alpha$, and α is an infinite cardinal such that $\alpha > |\operatorname{Mor}(\mathcal{C})|$.

Proof. Run the solution set argument of Lemma 3.7 for the set of κ-bounded cofibrations. Recall that the κ-bounded cofibrations generate the class of cofibrations. □

Say that a map $p : X \to Y$ is an L-*fibration* if it has the right lifting property with respect to all L-trivial cofibrations.

Observe that every L-fibration is a strict fibration, since S contains a generating set for the class of strict trivial cofibrations.

Lemma 6.8. *A map $p : X \to Y$ is an L-fibration and an L-equivalence if and only if p is a trivial strict fibration.*

Proof. The proof is the same as for Lemma 3.4. □

Theorem 6.9. *Suppose that S is a set of cofibrations which satisfies the list of Basic Assumptions above. Let the L-equivalences and L-fibrations be defined relative to the set S as above. Then with these definitions the category $\mathrm{Spt}_T(\mathcal{C})$ satisfies the axioms for a closed simplicial model category.*

Proof. The factorization axiom **CM5** follows from Lemmas 6.5, 6.7 and 6.8. The rest of the closed model axioms are trivial to verify.

For the simplicial model structure, we need to show that if $i : A \to B$ is a cofibration and an L-equivalence, then all maps

$$i \wedge \partial\Delta^n_+ : A \wedge \partial\Delta^n_+ \to B \wedge \partial\Delta^n_+$$

are L-equivalences. By replacing by a cofibrant model if necessary, it is enough to assume that A is cofibrant. Then one uses the usual patching argument for the category of cofibrant objects in the L-model structure for $\mathrm{Spt}_T(\mathcal{C})$ to compare pushouts of the form

$$\begin{array}{ccc} A \wedge \partial\Delta^{n-1}_+ & \longrightarrow & A \wedge \Lambda^n_{k+} \\ \downarrow & & \downarrow \\ A \wedge \Delta^{n-1}_+ & \longrightarrow & A \wedge \partial\Delta^n_+ \end{array}$$

to show inductively that the question reduces to showing that the map

$$i \wedge i : A \wedge A \to B \wedge B$$

is an L-equivalence. But $i \wedge i$ has the left lifting property with respect to all L-fibrations, and must therefore be an L-trivial cofibration. □

Lemma 6.10. *The L-structure on $\mathrm{Spt}_T(\mathcal{C})$ is left proper: given a pushout diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \xrightarrow[f_*]{} & D \end{array}$$

in which i is a cofibration, if f is an L-equivalence then f_* is an L-equivalence.

Proof. The original diagram may be replaced up to strict weak equivalence by a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f'} & C' \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \xrightarrow[f'_*]{} & D' \end{array}$$

in which f' is a cofibration and an L-equivalence. But then f'_* is also an L-trivial cofibration and is in particular an L-equivalence. □

We also have the following, which is a standard result in localization theory, with a standard proof:

Lemma 6.11. *Every injective object is L-fibrant, so that the L-fibrant T-spectra coincide with the injective T-spectra.*

Proof. Suppose that X is injective, and suppose given a diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\downarrow & & \\ B & & \end{array}$$

where the morphism i is a cofibration and an L-equivalence. Then $\alpha = \alpha' \cdot j$ for some map $\alpha' : LA \to X$ since X is injective, and so there is a diagram

$$\begin{array}{ccccc} A & \xrightarrow{j} & LA & \xrightarrow{\alpha'} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle Li} & & \\ B & \xrightarrow[j]{} & LB & & \end{array}$$

which factorizes the original. The map Li is a strict equivalence by Corollary 6.6, and one finishes the proof in the obvious way. □

Here's something else that follows from the general theory:

Lemma 6.12. *Suppose that $p : X \to Y$ is a strict fibration and that the maps $j : X \to LX$ and $j : Y \to LY$ are strict equivalences. Then p is an L-fibration.*

The proof is the same as that given for Lemma 3.9.

7 Stable homotopy theory of T-spectra

Here is how the basic applications of the localization result Theorem 6.9 arise.

Write J for the set of maps

$$\Sigma^\infty_T C[-n] \to \Sigma^\infty_T D[-n], \; n \geq 0,$$

which are induced by a set of cofibrations containing the α-bounded trivial cofibrations $C \to D$ of pointed simplicial presheaves.

Suppose that the set S of cofibrations is generated over J by the set of cofibrant replacements of the maps

$$\Sigma_T^\infty T[-1-n] \to S_T[-n].$$

In general such a set of cofibrations S gives a stable model structure for some localization of the category of pointed simplicial presheaves. For example, the maps in J could be induced by a set of generating trivial cofibrations for some f-local theory for some cofibration f. If J consists of nothing but the maps induced by the α-bounded trivial cofibrations of pointed simplicial presheaves, we are producing a "bare" stable model structure on $\mathrm{Spt}_T(\mathcal{C})$.

In all such cases, the L-equivalences and L-fibrations will be called *stable equivalences* and *stable fibrations*, respectively.

The examples to keep in mind are the T-spectrum objects $\mathrm{Spt}_T(Sm|_S)_{Nis}$ on the smooth Nisnevich site for a decent scheme S, where T is either the simplicial circle S^1 or the Tate object $S^1 \wedge \mathbb{G}_m$, and f is a choice of rational point $* \to \mathbb{A}^1$ on the affine line over S.

• The "bare" theory for $T = S^1$ is the stable structure for presheaves of spectra on $(Sm|_S)_{Nis}$ that we've already discussed.

• The f-local theory on $\mathrm{Spt}_T(Sm|_S)_{Nis}$ for $T = S^1 \wedge \mathbb{G}_m$ is the Morel-Voevodsky motivic stable model structure [17]. The f-local theory on the ordinary category $\mathrm{Spt}_{S^1}(Sm|_S)_{Nis}$ (aka. motivic S^1-spectra) is also important, in that it is a major technical device fore analyzing the former. By the same techniques, there is a motivic stable model structure for T-spectrum objects, for *all* pointed simplicial presheaves T (compare with [8]).

Here's the first thing that's special about these stable homotopy theories:

Lemma 7.1. *A map $p : X \to Y$ is an injective fibration if and only if p is a strict fibration and all diagrams of pointed simplicial presheaf maps*

$$\begin{array}{ccc} X^n & \xrightarrow{\sigma_*} & \Omega_T X^{n+1} \\ \downarrow & & \downarrow \\ Y^n & \xrightarrow[\sigma_*]{} & \Omega_T Y^{n+1} \end{array}$$

are homotopy cartesian.

Proof. This follows from the fact that the map p is an injective fibration if and only if p induces trivial fibrations of simplicial presheaves

$$\mathbf{Hom}(D, X) \to \mathbf{Hom}(C, X) \times_{\mathbf{Hom}(C,Y)} \mathbf{Hom}(C, X)$$

(internal function complexes) for all generators $C \to D$ of the set S. Apply this criterion to the two classes of generators for S. □

Corollary 7.2. *A T-spectrum X is stably fibrant if and only if all simplicial presheaves X^n are globally fibrant and all maps $X^n \to \Omega_T X^{n+1}$ are local weak equivalences.*

To go further, we need to make assumptions on the suspending object T.

In all that follows, $\Omega_T Y$ will be shorthand for the pointed simplicial presheaf $\mathbf{Hom}(T, Y)$ defined by

$$\mathbf{Hom}(T, Y)(U) = \mathbf{hom}(T|_U, Y|_U)$$

for $U \in \mathcal{C}$. This is tbe internal pointed function complex, also defined by the adjunction isomorphism

$$\hom(A \wedge T, Y) \cong \hom(A, \mathbf{Hom}(T, Y)).$$

Note that if K is a pointed simplicial set, then

$$\mathbf{Hom}(\Gamma^* K, Y)(U) = \mathbf{hom}(K, Y(U))$$

by adjointness. Similarly, if V is an S-scheme, then

$$\mathbf{Hom}(V_+, Y)(U) \cong Y(V \times U)$$

on $Sch|_S$ for all S-schemes U.

Say that T is *compact up to equivalence* if for any filtered diagram $i \mapsto X_i$ of globally fibrant pointed simplical presheaves the map

$$\varinjlim_i \Omega_T X_i \to \Omega_T F(\varinjlim_i X_i)$$

is a local weak equivalence.

Examples

1) Suppose that K is a pointed finite simplicial set. Then K is compact up to equivalence. In effect, the functor Ω_K commutes with filtered colimits, so one only has to show that Ω_K preserves local weak equivalences between presheaves of Kan complexes. But it's easy to see that Ω_K preserves local trivial fibrations, so a standard trick gives it.

2) Suppose that S is a decent scheme and that the category $Sch|_S$ has the Nisnevich topology. Then the pointed simplicial presheaf V_+ on $(Sch|_S)_{Nis}$ associated to every S-scheme V is compact up to equivalence. All Y_i take distinguished squares to homotopy cartesian diagrams, so that $\varinjlim_i Y_i$ also has this property. The Nisnevich descent theorem [17] implies that the globally fibrant model map

$$\varinjlim_i X_i \to F(\varinjlim_i X_i)$$

is a weak equivalence in each section, so that all maps

$$\varinjlim_i X_i(V \times U) \to F(\varinjlim_i X_i)(V \times U)$$

are weak equivalences. In particular, the map

$$\varinjlim_i \Omega_{V_+} X_i \to \Omega_{V_+} F(\varinjlim_i X_i)$$

is a sectionwise weak equivalence.

3) The same arguments imply that all finite simplicial sets K and all schemes V are compact up to equivalence in the motivic model structure on $Sch|_S$.

4) If T and T' are compact up to equivalence, then $T \wedge T'$ is compact up to equivalence.

It follows that the Tate object $S^1 \wedge \mathbb{G}_m$ is compact up to equivalence for both the Nisnevich local and motivic model structures on $Sch|_S$.

Exercise: $\mathbb{G}_m$ is pointed by the identity here, ie. it does not have a disjoint base point, so there's something extra to do to verify this last claim.

Starting with a T-spectrum X, define functors $X \mapsto Q^k X$ for $k \geq 0$ by specifying that

$$Q^0 X = FX$$

and

$$Q^{k+1} X = \Omega_T Q^k X[1]$$

(Ω_T = fake T-loops). The map $Q^k X \to Q^{k+1} X$ is the canonical map

$$Q^k X \xrightarrow{\sigma_*} \Omega_T Q^k X[1].$$

Set

$$QX = F(\varinjlim_k Q^k X).$$

Write $\eta : X \to QX$ for the natural composite

$$X \to FX = Q^0 X \to \varinjlim_k Q^k X \to F(\varinjlim_k Q^k X) = QX.$$

Lemma 7.3. *Suppose that T is compact up to equivalence. Then QX is stably fibrant.*

Proof. All objects $Q^k X$ are strictly fibrant. In the diagram

$$\begin{array}{ccccc}
\varinjlim_k Q^k X & & \xrightarrow{\quad j \quad} & & F(\varinjlim_k Q^k X) \\
{\scriptstyle \cong}\downarrow & \searrow^{\sigma_*} & & & \downarrow{\scriptstyle \sigma_*} \\
\varinjlim_k \Omega_T Q^k X[1] & \longrightarrow & \Omega_T(\varinjlim_k Q^k X)[1] & \longrightarrow & \Omega_T F(\varinjlim_k Q^k X)[1]
\end{array}$$

the indicated vertical map is an isomorphism by cofinality, and the bottom horizontal composite is a strict equivalence since T is compact up to equivalence. It follows that the vertical map σ_* is a strict equivalence. □

Now here's the theorem:

Theorem 7.4. *Suppose that T is compact up to equivalence. Then QX is stably fibrant, and the map $\eta : X \to QX$ is a stable equivalence.*

Proof. QX is stably fibrant. In the diagram

$$\begin{array}{ccc} X & \longrightarrow & QX \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j_*} \\ LX & \longrightarrow & QLX \end{array} \tag{1}$$

the map j_* is a strict equivalence because all pushouts $C \to D$ of generators $A \to B$ of S (remember S?) induces strict equivalences $QC \to QD$ (since each map $C \to D$ is an equivalence above a certain level). The map $\eta : LX \to QLX$ is a strict equivalence because LX is stably fibrant. It follows that $\eta : X \to QX$ is a stable equivalence. □

Corollary 7.5. *Suppose that T is compact up to equivalence. Then the stable model structure on the category of T-spectra is proper.*

Proof. Suppose given a pullback diagram

$$\begin{array}{ccc} Z \times_Y X & \xrightarrow{f_*} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[f]{} & Y \end{array}$$

of T-spectra such that p is a strict fibration and f is a stable equivalence. The induced diagram

$$\begin{array}{ccc} Q(Z \times_Y X) & \longrightarrow & QX \\ \downarrow & & \downarrow \\ QZ & \longrightarrow & QY \end{array}$$

is strictly homotopy cartesian, and the map $QZ \to QY$ is a strict equivalence. □

Here's the recognition principle for stable fibrations. It is the analog of Theorem 3.8.

Theorem 7.6. *Suppose that T is compact up to equivalence, and suppose that $p : X \to Y$ is a strict fibration. Then p is a stable fibration if and only if the diagram*

$$\begin{array}{ccc} X & \xrightarrow{j} & LX \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Lp} \\ Y & \xrightarrow[j]{} & LY \end{array} \tag{2}$$

is strictly homotopy cartesian.

Proof. Suppose that the diagram (2) is strictly homotopy cartesian. There is a factorization

$$\begin{array}{ccc} LX & \xrightarrow{i} & Z \\ & \searrow^{Lp} & \downarrow^{q} \\ & & LY \end{array}$$

of Lp such that i is a stable equivalence and q is an injective fibration. But then Z is injective, hence stably fibrant, so that i is a strict equivalence. It also follows from Lemma 6.12 that q is a stable fibration. By pulling back q along j, we see from the hypothesis that the induced map

$$X \to Y \times_{LY} Z$$

is a strict equivalence. Every trivial strict fibration is an stable fibration, and it follows that p is a retract of an stable fibration, and hence is itself a stable fibration.

Suppose that $p : X \to Y$ is a stable fibration. Find a factorization

$$\begin{array}{ccc} LX & \xrightarrow{i} & Z \\ & \searrow^{Lp} & \downarrow^{q} \\ & & LY \end{array}$$

such that i is a trivial stable cofibration and q is a stable fibration. Then i is a stable equivalence between stably fibrant objects, and is therefore a strict equivalence. In the pullback diagram

$$\begin{array}{ccc} Y \times_{LY} Z & \xrightarrow{pr} & Z \\ {\scriptstyle q_*}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \longrightarrow & LY \end{array}$$

the projection map pr is a stable equivalence since the stable model structure is proper by Corollary 7.5. It follows that the comparison

$$\begin{array}{ccc} X & \xrightarrow{\theta} & Y \times_{LY} Z \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q_*} & \\ & Y & \end{array}$$

is a stable equivalence between stable fibrations, and θ is therefore a strict equivalence by a standard argument. ☐

Remark 7.7. The assumption that T is compact up to equivalence in the statement of Theorem 7.6, is only required for showing that the diagram (2) is strictly homotopy cartesian if p is a stable fibration.

It is a basic property of ordinary stable homotopy theory that the suspension and loop are inverse to each other in that they form a Quillen equivalence Spt $\leftrightarrows$ Spt. There are various proofs of this in the literature, but secretly it depends on the fact that the cyclic permutation $(3,2,1)$ induces a map

$$(3,2,1) : S^3 \to S^3$$

(3-fold smash products) by permuting smash factors, and that this map is pointed homotopic to the identity. The latter is so because the map has degree 2.

Here's the general theorem:

Theorem 7.8. *Suppose that T is compact up to equivalence. Suppose that X is a T-spectrum, and let $j : X \wedge T \to L(X \wedge T)$ be the natural stable fibrant model for $X \wedge T$. Suppose that the map*

$$(3,2,1) : T^3 \to T^3$$

represents the identity in the (f-local) pointed homotopy category. Then the composite

$$X \xrightarrow{\eta} \Omega_T(X \wedge T) \xrightarrow{\Omega_T j} \Omega_T L(X \wedge T)$$

is a stable equivalence.

The proof [8, Theorem 3.11] is a somewhat delicate layer filtration argument, which reduces to the case $X = \Sigma^\infty_T K$ for a pointed simplicial presheaf K.

In global sections, the homotopy groups $\pi_r LY^n$ are computed as the filtered colimit

$$[S^r, Y^n] \xrightarrow{\Sigma} [T \wedge S^r, Y^{n+1}] \xrightarrow{\Sigma} \dots,$$

where Σ takes a map $\theta : S^r \to Y^n$ to the composite

$$T \wedge S^r \xrightarrow{T \wedge \theta} T \wedge Y^n \xrightarrow{\sigma} Y^{n+1}.$$

This follows from Theorem 7.4.

When $Y = \Sigma^\infty_T K$, the map Σ is smashing with T on the left. The composite

$$[T^k \wedge S^r, Y^{n+k}] \to [T^k \wedge S^r, \Omega_T L(Y \wedge T)^{n+k}] \cong [T^k \wedge S^r \wedge T, Y^{n+k} \wedge T]$$

is smashing with T on the right. The proof is finished by using the comparison diagram

$$\begin{array}{ccccc}
[T^k \wedge S^r, T^{n+k} \wedge K] & \xrightarrow{T^2 \wedge} & [T^{2+k} \wedge S^r, T^{2+n+k} \wedge K] & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \wedge T} & & \downarrow{\scriptstyle \wedge T} & & \\
[T^k \wedge S^r \wedge T, T^{n+k} \wedge K \wedge T] & \xrightarrow{T^2 \wedge} & [T^{2+k} \wedge S^r \wedge T, T^{2+n+k} \wedge K \wedge T] & \longrightarrow & \cdots \\
{\scriptstyle c_t}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle c_t} & & \\
[T^{1+k} \wedge S^r, T^{1+n+k} \wedge K] & \xrightarrow[T^2 \wedge]{} & [T^{3+k} \wedge S^r, T^{3+n+k} \wedge K] & \longrightarrow & \cdots
\end{array}$$

Here, c_t is a conjugation isomorphism defined by twisting, and the bottom square commutes by the hypotheses on T^3, and the vertical composites are instances of $T\wedge$.

Corollary 7.9. *Under the hypotheses of Theorem 7.8, if Y is stably fibrant, then the canonical (evaluation) map $\epsilon : \Omega_T Y \wedge T \to Y$ is a stable equivalence.*

Proof. Let $j : \Omega_T Y \wedge T \to L(\Omega_T Y \wedge T)$ be a stably fibrant model, and extend ϵ to a map $\epsilon_* : L(\Omega_T Y \wedge T) \to Y$. Form the diagram

$$\begin{array}{ccccc} \Omega_T Y & \xrightarrow{\eta} & \Omega_T(\Omega_T Y \wedge T) & \xrightarrow{\Omega_T j} & \Omega_T L(\Omega_T Y \wedge T) \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \Omega_T \epsilon} & \swarrow{\scriptstyle \Omega_T \epsilon_*} & \\ & & \Omega_T Y & & \end{array}$$

Then $\Omega_T \epsilon_*$ is a stable equivalence by Theorem 7.8, and so ϵ_* is a stable equivalence by a calculation. □

The Tate object T in motivic homotopy theory is the most prominent example of an object satisfying the conditions of Theorem 7.8.

Lemma 7.10 (Voevodsky)**.** *The cyclic permutation $(3, 2, 1) \in \Sigma_3$ acts as the identity on T^3 in the pointed motivic homotopy category, where T is the Tate object $T = S^1 \wedge \mathbb{G}_m$.*

Proof. There is an identification

$$T^3 \simeq \mathbb{A}^3/(\mathbb{A}^3 - 0)$$

in the motivic model category, and the action of Σ_3 is the restriction of a pointed algebraic group action

$$Gl_3 \times T^3 \to T^3.$$

The permutation matrix $(3, 2, 1)$ is a product of elementary transformation matrices, and so there is an algebraic path

$$\omega : \mathbb{A}^1 \to Gl_3$$

from the identity matrix to $(3, 2, 1)$. The composite

$$\mathbb{A}^1 \times T^3 \to Gl_3 \times T^3 \to T^3$$

gives a pointed homotopy from $(3, 2, 1) : T^3 \to T^3$ to the identity. □

8 $(S^1 \wedge K)$-spectra

Suppose that the pointed simplicial presheaf K is compact up to equivalence. The class of pointed simplicial presheaves which are compact up to equivalence is closed under finite smash products and includes all finite pointed simplicial sets. It follows that $S^1 \wedge K$ is compact up to equivalence, so all results of the previous section apply to $(S^1 \wedge K)$-spectra. I shall assume that K is compact up to equivalence throughout this section.

In what follows it's best to think of the bonding maps for an $(S^1 \wedge K)$-spectrum X as maps of the form

$$\sigma : S^1 \wedge X^n \wedge K \to X^{n+1}.$$

These morphisms induce maps

$$\begin{aligned}\sigma_* : &S^k \wedge X^n \wedge K^r = \\ &S^{k-1} \wedge S^1 \wedge X^n \wedge K \wedge K^{r-1} \xrightarrow{S^{k-1}\wedge\sigma\wedge K^{r-1}} S^{k-1} \wedge X^{n+1} \wedge K^{r-1}\end{aligned}$$

in the obvious way.

An $(S^1 \wedge K)$-spectrum X determines a K-spectrum object $X^{*,*}$ in spectra, which at K-level n is the spectrum

$$\begin{aligned}X^{n,*} :\ &X^0 \wedge K^n, X^1 \wedge K^{n-1}, \dots, \\ &X^{n-1} \wedge K, X^n, S^1 \wedge X^n, S^2 \wedge X^n, \dots\end{aligned}$$

The bonding maps for $X^{n,*}$ are the maps

$$\sigma_* : S^1 \wedge X^j \wedge K^{n-j} \to X^{j+1} \wedge K^{n-j-1}$$

up to level $n-1$, and are identities defined by smashing with S^1 beyond. The K-bonding maps

$$X^{n,*} \wedge K \to X^{n+1,*}$$

are identity maps defined by smashing with K up to level n and are instances of σ_* in levels $n+1$ and above. The fact that one actually does get a map of S^1-spectra this way is essentially a consequence of the fact that the morphisms σ_* respect smashing with S^1 and K.

An $(S^1 \wedge K)$-spectrum X has bigraded presheaves of stable homotopy groups $\pi_{s,t}X$, defined by

$$\pi_{s,t}X(U) = \varinjlim_{n\geq 0} [S^{n+s} \wedge K^{n+t}|_U, X^n|_U]$$

where the homotopy classes of maps are computed for pointed simplicial presheaves on $\mathcal{C}/U$ (= $\mathrm{Sch}|_U$ usually), and the transition maps are defined by suspension in the "obvious" way: a representing map

$$\alpha : S^{n+s} \wedge K^{n+t} \to X^n$$

is sent to the composite

$$S^{n+1+s} \wedge K^{n+1+t} \xrightarrow{S^1\wedge\alpha\wedge K} S^1 \wedge X^n \wedge K \xrightarrow{\sigma} X^{n+1}.$$

In $\pi_{s,t}X$, the index s is often called the *degree*, while t is the *weight*.

These stable homotopy group presheaves are specializations of bigraded stable homotopy groups $\pi_{s,t}Y$ which are defined for K-spectrum objects Y in spectra by

$$\pi_{s,t}Y(U) = \varinjlim_{k,l} [S^{k+s} \wedge K^{l+t}, Y^{k,l}]_U,$$

where the notation indicates that the homotopy classes are computed over $U \in \mathcal{C}$. This means that there are natural isomorphisms

$$\pi_{s,t}Y \cong \pi_{s,t}dY$$

of presheaves for all K-spectrum objects Y by a cofinality argument. Here, dY is the $(S^1 \wedge K)$-spectrum with $dY^n = Y^{n,n}$ and with bonding maps given by the composites

$$S^1 \wedge Y^{n,n} \wedge K \to S^1 \wedge Y^{n+1,n} \to Y^{n+1,n+1}.$$

It follows that there are natural isomorphisms

$$\pi_{s,t}X^{*,*} \cong \pi_{s,t}X$$

for all $(S^1 \wedge K)$-spectra X.

The bonding maps $Y^n \wedge K \to Y^{n+1}$ in a K-spectrum object Y induce homomorphisms of stable homotopy classes of maps

$$\begin{aligned} &[S[s] \wedge K^{n+t}, Y^n]_U \to [S[s] \wedge K^{n+t+1}, Y^{n+1}]_U \\ &\to [S[s] \wedge K^{n+t+2}, Y^{n+2}]_U \to \dots \end{aligned}$$

for all $U \in \mathcal{C}$, and the filtered colimit of the system is $\pi_{s,t}Y(U)$.

Note that there are isomorphisms of presheaves

$$\pi_k QX^n(U) \cong \pi_{k-n,-n}X(U).$$

Then we have the following:

Lemma 8.1. *A map $f : X \to Y$ is a stable equivalence of $(S^1 \wedge K)$-spectra if and only if it induces isomorphisms of presheaves*

$$\pi_{s,t}X \cong \pi_{s,t}Y$$

for all integers s and t.

Suppose that

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a strict fibre sequence of $(S^1 \wedge K)$-spectra. Every map $f : Z \to W$ of K-spectrum objects has a factorization

$$\begin{array}{ccc} Z & \xrightarrow{j} & V \\ & \searrow^{f} & \downarrow{q} \\ & & W \end{array}$$

where q is a strict fibration in each K-level and j is a cofibration and a strict weak equivalence in each K-level. Take such a factorization

$$\begin{array}{ccc} X^{*,*} & \xrightarrow{j} & V \\ & \searrow^{p} & \downarrow q \\ & & Y^{*,*} \end{array}$$

for the map induced by the T-spectrum map $p : X \to Y$, and let $\overline{F}$ be the fibre of q. Then there are induced comparisons of fibre sequences of simplicial presheaves

$$\begin{array}{ccccc} F^n & \xrightarrow{i} & X^n & \xrightarrow{p} & Y^n \\ \downarrow & & \downarrow \simeq & & \downarrow \cong \\ \overline{F}^{n,n} & \longrightarrow & V^{n,n} & \longrightarrow & Y^{n,n} \end{array} \tag{3}$$

for each $n \geq 0$, and it follows (by properness for pointed simplicial presheaves) that the induced map $F \to d\overline{F}$ is a strict weak equivalence.

Suppose that

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a level strict fibre sequence of K-spectrum objects and that Y is strictly fibrant in all K-levels. Then all induced sequences

$$\Omega_K^{t+n} F^n \to \Omega_K^{t+n} X^n \to \Omega_K^{t+n} Y^n$$

are strict fibre sequences of presheaves of spectra, and all spectra $\Omega_K^{t+n} Y^n$ are strictly fibrant. It follows that there is a long exact sequence in presheaves of stable homotopy groups of the form

$$\begin{aligned} \dots \to \pi_s \Omega_K^{t+n} F^n \to \pi_s \Omega_K^{t+n} X^n \to \pi_s \Omega_K^{t+n} Y^n \\ \xrightarrow{\partial} \pi_{s-1} \Omega_K^{t+n} F^n \to \dots \end{aligned}$$

There are, as well, comparisons of fibre sequences

$$\begin{array}{ccccc} \Omega_K^{t+n} F^n & \longrightarrow & \Omega_K^{t+n} X^n & \longrightarrow & \Omega_K^{t+n} Y^n \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_K^{t+n+1} F^{n+1} & \longrightarrow & \Omega_K^{t+n+1} X^{n+1} & \longrightarrow & \Omega_K^{t+n+1} Y^{n+1} \end{array}$$

induced by the respective K-spectrum object structures. Thus taking a filtered colimit in n gives a long exact sequence

$$\dots \to \pi_{s,t} F \xrightarrow{i_*} \pi_{s,t} X \xrightarrow{p_*} \pi_{s,t} Y \xrightarrow{\partial} \pi_{s-1,t} F \to \dots \tag{4}$$

in presheaves of bigraded stable homotopy groups. Note that the degree s changes while the weight t does not.

It follows from the remarks above (specifically, the existence of diagram (3)) that there is a natural long exact sequence of the form (4) for any strict fibre sequence

$$F \to X \to Y$$

of $(S^1 \wedge K)$-spectra.

Strict fibre and cofibre sequences coincide up to natural stable equivalence in $(S^1 \wedge K)$-spectra. The proof comes in three parts:

Lemma 8.2. *Suppose that $p : X \to Y$ is a strict fibration of $(S^1 \wedge K)$-spectra, with fibre F. Then the canonical map*

$$X/F \to Y$$

is a stable equivalence.

Proof. The Lemma follows from the corresponding result for presheaves of spectra, by replacing the given fibre sequence by a fibre sequence of K-spectrum objects in spectra. □

Lemma 8.3. *Suppose that*

$$\begin{array}{ccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 \end{array}$$

is a comparison of level cofibre sequences of $(S^1 \wedge K)$-spectra. If any two of the maps f_1, f_2 and f_3 are stable equivalences, then so is the third.

Proof. It suffices to assume that all objects are cofibrant.

The comparison diagram in the statement induces a comparison of fibre sequences

$$\begin{array}{ccccc} \mathbf{hom}(B_3, Z) & \longrightarrow & \mathbf{hom}(B_2, Z) & \longrightarrow & \mathbf{hom}(B_1, Z) \\ {\scriptstyle f_3^*}\downarrow & & \downarrow{\scriptstyle f_2^*} & & \downarrow{\scriptstyle f_1^*} \\ \mathbf{hom}(A_3, Z) & \longrightarrow & \mathbf{hom}(A_2, Z) & \longrightarrow & \mathbf{hom}(A_1, Z) \end{array}$$

for all stably fibrant objects Z. There are stable equivalences

$$\Omega_T LB \wedge K \wedge S^1 \cong \Omega_T LB \wedge T \xrightarrow{\epsilon} LB$$

(Corollary 7.9) so that the comparison of fibre sequences is a comparison of fibre sequences of infinite loop spaces. Thus if any two of the vertical maps are (stable) equivalences, then so is the third. □

Lemma 8.4. *Suppose that $i : A \to B$ is a level cofibration of $(S^1 \wedge T)$-spectra, and take a factorization*

$$\begin{array}{ccc} B & \xrightarrow{j} & Z \\ & {\scriptstyle\pi}\searrow & \downarrow{\scriptstyle p} \\ & & B/A \end{array}$$

of the quotient map $\pi : B \to B/A$, where j is a strict trivial cofibration and p is a fibration. Let F be the fibre of p. Then the induced map $A \to F$ is a stable equivalence.

Proof. The canonical map $p_* : Z/F \to B/A$ associated to the fibration $p : Z \to B/A$ is a stable equivalence by Lemma 8.2. There is also a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & B/A \\ \downarrow & & \simeq \downarrow {\scriptstyle j} & & \downarrow {\scriptstyle j_*} \\ F & \longrightarrow & Z & \longrightarrow & Z/F \end{array}$$

But $p_* j_* = 1$ so that j_* is a stable equivalence. It follows that the map $A \to F$ in the diagram (which is the map of interest) is a stable equivalence, by Lemma 8.3. □

Corollary 8.5. *Every level cofibre sequence*

$$A \to B \to B/A$$

has a naturally associated long exact sequence

$$\dots \pi_{s,t} A \to \pi_{s,t} B \to \pi_{s,t}(B/A) \xrightarrow{\partial} \pi_{s-1,t} A \to \dots.$$

of presheaves of stable homotopy groups.

Corollary 8.6. *There are natural isomorphisms*

$$\pi_{s+1,t}(Y \wedge S^1) \cong \pi_{s,t} Y$$

for all $(S^1 \wedge K)$-spectra Y.

Corollary 8.7 (additivity)**.** *Suppose that X and Y are $(S^1 \wedge K)$-spectra. Then the canonical map*

$$c : X \vee Y \to X \times Y$$

is a stable equivalence.

9 Symmetric T-spectra

For now, $\mathcal{C}$ will be an arbitrary small Grothendieck site.

A *symmetric space* X consists of pointed simplicial presheaves X^n, $n \geq 0$ on $\mathcal{C}$ with symmetric group actions

$$\Sigma_n \times X^n \to X^n$$

A morphism $f : X \to Y$ consists of pointed simplicial presheaf morphisms $X^n \to Y^n$, $n \geq 0$, which respect the symmetric group actions. The category of symmetric spaces will be denoted (following [4], see also [9]) by $s_* \operatorname{Pre}(\mathcal{C})^{\Sigma}$.

Example 9.1. Suppose that T is a pointed simplicial presheaf. Then the sequence

$$S^0,\ T,\ T \wedge T,\ T^3,\ \dots$$

forms a symmetric space, which will be denoted by S_T.

If K is a pointed simplicial presheaf, smashing S_T with K gives the suspension object

$$\Sigma_T^\infty K = S_T \wedge K.$$

Given symmetric spaces X and Y, their *tensor product* $X \otimes Y$ is specified in degree n by

$$(X \otimes Y)^n = \bigvee_{r+s=n} \Sigma_n \otimes_{\Sigma_r \times \Sigma_s} (X^r \wedge Y^s).$$

A map of symmetric spaces $X \otimes Y \to Z$ consists of $(\Sigma_r \times \Sigma_s)$-equivariant maps

$$X^r \wedge Y^s \to Z^{r+s}$$

for all $r, s \geq 0$.

Examples

1) The *canonical map*

$$\otimes : S_T \otimes S_T \to S_T$$

consists of the canonical isomorphisms

$$S^r \wedge S^s \cong S^{r+s}$$

2) Write $c_{r,s} \in \Sigma_{r+s}$ for the shuffle which is defined by

$$c_{r,s}(i) = \begin{cases} s+i & i \leq r \\ i-r & i > r \end{cases}$$

The *twist automorphism*

$$\tau : X \otimes Y \to Y \otimes X$$

is uniquely determined by the composites

$$X^r \wedge Y^s \xrightarrow{\tau} Y^r \wedge X^s \to (Y \otimes X)^{r+s} \xrightarrow{c_{s,r}} (Y \otimes X)^{r+s}.$$

(we have to multiply by the shuffle $c_{s,r}$ to make the composite $(\Sigma_r \times \Sigma_s)$-equivariant). Then the composites

$$X \otimes Y \xrightarrow{\tau} Y \otimes X \xrightarrow{\tau} X \otimes Y$$

are identities.

The tensor product $(X, Y) \mapsto X \otimes Y$ is symmetric monoidal. The map $\otimes : S_T \otimes S_T \to S_T$ gives S_T the structure of an abelian monoid in the category of symmetric spaces.

A *symmetric T-spectrum* X is a symmetric space with the structure

$$m_X : S_T \otimes X \to X$$

of a module over S_T. This means that X comes equipped with (bonding) maps

$$\sigma_{1,s} : T \wedge X^s \to X^{1+s}$$

such that all composite bonding maps

$$T^r \wedge X^s \to X^{r+s}$$

are equivariant for the inclusion $\Sigma^r \times \Sigma^s \subset \Sigma^{r+s}$. There is an obvious category of such things, which we denote by $\mathrm{Spt}_T^{\Sigma}(\mathcal{C})$.

Example 9.2. Certainly the symmetric spaces S_T and $\Sigma_T^{\infty} K = S_T \wedge K$ have the structure of symmetric spectra.

More generally, if Y is a symmetric space, then $S_T \otimes Y$ is a symmetric T-spectrum: it is the free symmetric T- spectrum on Y in an obvious sense.

Example 9.3. Write Γ for the category of finite pointed sets and pointed functions between them. A Γ-*space* is a functor $A : \Gamma \to s_* \operatorname{Pre}(\mathcal{C})$ defined on the category of finite pointed sets and taking values in pointed simplicial presheaves. The finite pointed sets K, L determine a canonical map

$$K \wedge A(L) \to A(K \wedge L)$$

so it follows that there are bisimplicial object maps

$$S^k \wedge A(S^n) \to A(S^{k+n}),$$

which induce pointed simplicial presheaf maps

$$S^k \wedge dA(S^n) \to dA(S^{k+n}), \tag{5}$$

where d is the diagonal functor. It follows that the sequence

$$dA(S^0),\ dA(S^1),\ dA(S^2),\ \ldots$$

has the structure of a symmetric S^1-spectrum (aka. symmetric spectrum).

Γ-spaces are a common source of symmetric spectra, since every symmetric monoidal category determines a Γ-space, and hence a spectrum. The sphere spectrum S and all algebraic K-theory spectra are examples. Note that spectra which arise from Γ-spaces, according to the recipe given above are all connective [1].

Example 9.4. If K is a pointed simplicial presheaf and $n \geq 0$ there is a symmetric space G_nK with

$$(G_nK)^r = \begin{cases} * & r \neq n \\ \Sigma_n \otimes K = \bigvee_{\Sigma_n} K & r = n. \end{cases}$$

If we then define F_nK by

$$F_nK = S_T \otimes G_nK,$$

then there is a natural bijection

$$\hom_{\mathrm{Spt}_T^{\Sigma}(\mathcal{C})}(F_nK, Z) \cong \hom_{s_*\,\mathrm{Pre}(\mathcal{C})}(K, Z^n).$$

for all $n \geq 0$, pointed simplicial presheaves K and symmetric T-spectra Z. Note that $F_0K = \Sigma_T^{\infty}K$.

Here's why we care about symmetric spectra: the category of symmetric T-spectra has a symmetric monoidal smash product. Given symmetric T-spectra X, Y, the *smash product* $X \wedge Y$ is defined by the coequalizer

$$S_T \otimes X \otimes Y \rightrightarrows X \otimes Y \to X \wedge Y.$$

where the arrows $S_T \otimes X \otimes Y \rightrightarrows X \otimes Y$ are $m_X \otimes Y$ and the composite

$$S_T \otimes X \otimes Y \xrightarrow{\tau \otimes Y} X \otimes S_T \otimes Y \xrightarrow{X \otimes m_Y} X \otimes Y.$$

The relationship with T-spectra is mediated by a forgetful functor

$$U : \mathrm{Spt}_T^{\Sigma}(\mathcal{C}) \to \mathrm{Spt}_T(\mathcal{C})$$

which forgets the symmetric group actions.

I claim that U has a left adjoint

$$V : \mathrm{Spt}_T(\mathcal{C}) \to \mathrm{Spt}_T^{\Sigma}(\mathcal{C}).$$

The functor V is constructed inductively, by using the layer filtration L_nX of a T-spectrum X. In effect, for shifted suspension spectra $\Sigma_T^{\infty}K[-n]$, it must be the case that

$$\begin{aligned}\hom_{\mathrm{Spt}_T^{\Sigma}(\mathcal{C})}(V\Sigma_T^{\infty}K[-n], Z) &\cong \hom_{\mathrm{Spt}_T(\mathcal{C})}(\Sigma_T^{\infty}K[-n], UZ) \\ &\cong \hom_{s_*\,\mathrm{Pre}(\mathcal{C})}(K, Z^n)\end{aligned}$$

for V to be a left adjoint, so that there must be natural isomorphisms

$$V\Sigma_T^\infty K[-n] \cong F_n K.$$

The layer filtration $L_nX \subset X$ of a T-spectrum X is constructed by a sequence of pushouts

$$\begin{array}{ccc} \Sigma_T^\infty(T \wedge X^n)[-n-1] & \longrightarrow & L_nX \\ \downarrow & & \downarrow \\ \Sigma_T^\infty X^{n+1}[-n-1] & \longrightarrow & L_{n+1}X \end{array}$$

and so $VL_{n+1}X$ can be inductively specified by the pushouts

$$\begin{array}{ccc} F_{n+1}(T \wedge X^n) & \longrightarrow & VL_nX \\ \downarrow & & \downarrow \\ F_{n+1}X^{n+1} & \longrightarrow & VL_{n+1}X \end{array}$$

Write $VX = \varinjlim_n VL_nX$.

10 Model structures for symmetric spectra

A map $f : X \to Y$ of symmetric T-spectra is said to be a *level weak equivalence* if all component maps $X^n \to Y^n$ are local weak equivalences of simplicial presheaves. There are two basic model structures on the category of symmetric T-spectra for which the level equivalences are the weak equivalences.

There is a *projective structure* for which the fibrations are the levelwise global fibrations and the weak equivalences are the levelwise weak equivalences. This is in fact a cofibrantly generated closed simplicial model structure in an essentially obvious way, and the reader can produce it for an exercise. The function complex $\mathbf{hom}(X, Y)$ which is used to define the simplicial model structure has the definition that you might expect: its n-simplices are the maps $X \wedge \Delta^n_+ \to Y$.

It is easy to see that the functor V takes cofibrations (respectively strictly trivial cofibrations) of T-spectra to projective cofibrations (respectively trivial projective cofibrations) of symmetric T-spectra.

"Dually", an *injective fibration* is a map which has the right lifting property with respect to all level trivial cofibrations (ie. maps which are level cofibrations and level equivalences).

Proposition 10.1. *The level equivalences, level cofibrations and injective fibrations together give the category $\mathrm{Spt}_T^\Sigma(\mathcal{C})$ of symmetric T-spectra the structure of a proper closed simplicial model category. This model structure is cofibrantly generated, by the α-bounded level cofibrations and the α-bounded level trivial cofibrations.*

The model structure for symmetric T-spectra of Proposition 10.1 is the *injective model structure*. It's a little trickier to put in place than the projective structure (see also [9, Theorem 2], [8, Theorem 4.2]).

The major steps in a somewhat different proof involve the establishment of a bounded cofibration condition

Lemma 10.2. *Suppose that α is an infinite cardinal such that $\alpha > |\mathcal{C}|$. Suppose given level cofibrations*

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle i} \\ A & \longrightarrow & Y \end{array}$$

such that i is a level equivalence and the object A is α-bounded. Then there is an α-bounded subobject $B \subset Y$ with $A \subset B$ such that the induced map $B \cap X \to B$ is a level equivalence.

(the proof is essentially the same as that of the corresponding result for simplicial presheaves), and an invocation of the solution set trick

Lemma 10.3. *Suppose that $p : X \to Y$ has the right lifting property with respect to all α-bounded level trivial cofibrations and that p is a level equivalence. Then p has the right lifting property with respect to all cofibrations.*

Then Lemma 10.3 implies that every map $p : X \to Y$ of symmetric T-spectra which has the right lifting property with respect to all α-bounded level trivial cofibrations must be an injective fibration.

It's easy to see that if $p : Z \to W$ has the right lifting property with respect to all cofibrations then it is a trivial fibration.

One uses a transfinite induction and the bounded cofibration condition to show that a map $\pi : Z \to W$ has the right lifting property with respect to all level cofibrations (respectively level trivial cofibrations) if and only if it has the right lifting property with respect to all α-bounded level cofibrations (respectively α-bounded level trivial cofibrations).

It then follows from Lemma 10.3 that $\pi : Z \to W$ is a trivial fibration if and only if it has the right lifting property with respect to all α-bounded cofibrations.
Basic Assumptions: Suppose that S is a set of level cofibrations of symmetric T-spectra which includes the set J of α-bounded level trivial cofibrations. Suppose that all induced maps

$$(A \wedge D) \cup (B \wedge C) \to B \wedge D$$

are in S for each α-bounded cofibration $C \to D$ of pointed simplicial presheaves.

Suppose that γ is a cardinal such that $\gamma > |\operatorname{Mor}(\mathcal{C})|$, and that $\gamma > |B|$ for all morphisms $i : A \to B$ appearing in the set S. Suppose also that $\gamma > |S|$. Choose a cardinal λ such that $\lambda > 2^{\gamma}$.

Suppose that $f : X \to Y$ is a morphism of symmetric T-spectra. Just as for T-spectra, define a functorial system of factorizations

$$\begin{array}{ccc} X & \xrightarrow{i_s} & E_s(f) \\ & \searrow^{f} & \downarrow f_s \\ & & Y \end{array}$$

of the map f indexed on all ordinal numbers $s < \lambda$ as follows:

1) Given the factorization (f_s, i_s) define the factorization (f_{s+1}, i_{s+1}) by requiring that the diagram

$$\begin{array}{ccc} \bigvee_{\mathbf{D}} A & \xrightarrow{(\alpha_{\mathbf{D}})} & E_s(f) \\ {\scriptstyle \vee i}\downarrow & & \downarrow \\ \bigvee_{\mathbf{D}} B & \longrightarrow & E_{s+1}(f) \end{array}$$

is a pushout, where the wedge is indexed over all diagrams $\mathbf{D}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{\alpha_{\mathbf{D}}} & E_s(f) \\ {\scriptstyle i}\downarrow & & \downarrow {\scriptstyle f_s} \\ B & \xrightarrow[\beta_{\mathbf{D}}]{} & Y \end{array}$$

with $i : A \to B$ in the set S. Then the map i_{s+1} is the composite

$$X \xrightarrow{i_s} E_s(f) \xrightarrow{g_*} E_{s+1}(f)$$

2) If s is a limit ordinal, set $E_s(f) = \varinjlim_{t<s} E_s(f)$.

Set $E_\lambda(f) = \varinjlim_{s<\lambda} E_s(f)$. Then there is an induced factorization

$$\begin{array}{ccc} X & \xrightarrow{i_\lambda} & E_\lambda(f) \\ & \searrow^{f} & \downarrow f_\lambda \\ & & Y \end{array}$$

of the map f. Then i_λ is a level cofibration. The map f_λ has the right lifting property with respect to the cofibrations $i : A \to B$ in S, since any map $\alpha : A \to E_\lambda(f)$ must factor through some $E_s(f)$ by the choice of cardinal λ.

Write $L(X) = E_\lambda(c)$ for the result of this construction when applied to the canonical map $c : X \to *$. Then we have the following set theoretic result:

Lemma 10.4. *1) Suppose that $t \mapsto X_t$ is a diagram of level cofibrations indexed by any cardinal $\gamma > 2^\alpha$. Then the natural map*

$$\varinjlim_{t<\gamma} L(X_t) \to L(\varinjlim_{t<\gamma} X_t)$$

is an isomorphism.

2) *The functor $X \mapsto L(X)$ preserves level cofibrations.*

3) *Suppose that ζ is a cardinal with $\zeta > \alpha$, and let $\mathcal{F}_\zeta(X)$ denote the filtered system of subobjects of X having cardinality less than ζ. Then the natural map*

$$\varinjlim_{Y \in \mathcal{F}_\zeta(X)} L(Y) \to L(X)$$

is an isomorphism.

4) *If $|X| < 2^\omega$ where $\omega \geq \alpha$ then $|L(X)| < 2^\omega$.*

5) *Suppose that U, V are subobjects of a presheaf of spectra X. Then the natural map*

$$L(U \cap V) \to L(U) \cap L(V)$$

is an isomorphism.

A map is said to be *injective* if it has the right lifting property with respect to all members of S, and an object X is *injective* if the map $X \to *$ is injective.

Note that all injective objects are fibrant for the injective model structure of Proposition 10.1. By construction, LX is injective.

Say that a morphism $f : X \to Y$ of $\mathrm{Spt}_T^\Sigma(C)$ is an *L-equivalence* if it induces a weak equivalence

$$f^* : \mathbf{hom}(Y, Z) \to \mathbf{hom}(X, Z)$$

of simplicial sets for all injective objects Z.

Every level equivalence (ie. equivalence for the injective structure) is an L-equivalence, and one can show, by analogy with the case of T-spectra that a map $X \to Y$ is an L-equivalence if and only if the induced map $LX \to LY$ is a level equivalence. In fact, the whole localization argument that we used for T-spectra just goes through for symmetric T-spectra as well, giving the general localization theorem for symmetric T-spectra:

Theorem 10.5. *The category $\mathrm{Spt}_T^\Sigma(\mathcal{C})$ of symmetric T-spectra, with the classes of cofibrations, L-equivalences and L-fibrations, satisfies the axioms for a left proper closed simplicial model category.*

The proof is written up in Lecture 010 of [13].

There is also the formal consequence of the constructions that we constantly exploit:

Lemma 10.6. *A symmetric T-spectrum X is L-fibrant if and only if it is injective.*

The model structure of Theorem 10.5 is also left proper in general. As one would have been led to expect from the corresponding construction for T-spectra, right properness is more of an issue.

11 Stable homotopy theory of symmetric T-spectra

Suppose that $f : A \to B$ is a member of a set of cofibrations of pointed simplicial presheaves (which could be empty), and that I is a set of generating cofibrations for the corresponding local model structure on $s_* \operatorname{Pre}(\mathcal{C})$. By a standard abuse, I shall call this the f-local structure. Suppose that I contains all α-bounded locally trivial cofibrations. Let J be the set of all level cofibrations

$$F_n C \to F_n D, \ n \geq 0,$$

which are induced by maps $C \to D$ appearing in the set J.

The f-local stable model structure for symmetric T-spectra (and the corresponding stable homotopy category) arises via Theorem 10.5 by formally inverting the set S of level cofibrations which is generated over J by the set of cofibrations obtained by applying the functor V to the cofibrant replacements of the maps

$$\Sigma_T^\infty T[-1-n] \to S_T[-n],$$

where S_T is the sphere spectrum for the T-spectrum category.

Say that an L-equivalence for this theory is a *stable equivalence*, and that an L-fibration is a *stable fibration*. Recall that the cofibrations for this theory are the level cofibrations.

Remark 11.1. There is a corresponding f-local model structure for the category $\operatorname{Spt}_T(\mathcal{C})$ of T-spectra, by the methods of Section 6. The weak equivalences and fibrations for that theory will be called stable equivalences and stable fibrations, respectively.

Remark 11.2. Theorem 10.5 also implies that there is a model structure on $\operatorname{Spt}_T^\Sigma(\mathcal{C})$ which arises by formally inverting the cofibrations in the set I. The corresponding model structure will be called the f-injective structure. The fibrant objects for this theory are the injective objects X such that all X^n are f-local. The weak equivalences are level f-equivalences — one sees this by showing that the functors F_n take f-local equivalences to level f-local equivalences.

By Lemma 10.6, a symmetric T-spectrum Z is stably fibrant if and only if it is S-injective, which means precisely that

- Z is injective,
- the underlying T-spectrum UZ is stably fibrant, meaning that all Z^n are f-local and all maps $Z^n \to \Omega_T Z^{n+1}$ are f-local (hence sectionwise) weak equivalences — see Lemma 7.1.

A formal argument (see Lemma 3 of [9]) implies that V takes cofibrations to level cofibrations. A left properness argument which starts with the layer filtration implies that V preserves level equivalences. An adjunction argument then implies that every (f-local) stable equivalence $A \to B$ of T-spectra induces a stable equivalence $VA \to VB$ of symmetric T-spectra.

Since the stable model structure for T-spectra is cofibrantly generated, every map $f : X \to Y$ of symmetric T-spectra has a natural factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & X_s \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p_s} \\ & & Y \end{array}$$

where i is a stably trivial cofibration and Up_s is a stable fibration. Applying this construction to the map $X \to *$ determines a natural stably trivial cofibration $i : X \to X_s$ such that UX_s is stably fibrant. Finally, consider the composite

$$X \xrightarrow{i} X_s \xrightarrow{j} IX_s$$

where $j : X_s \to IX_s$ is the natural f-injective model. Then IX_s is injective and the map $j : X_s \to IX_s$ is a level equivalence so that UIX_s is stably fibrant. The composite ji is a stable equivalence, and therefore determines a natural stably fibrant model construction the category of symmetric T-spectra.

It follows that a map $f : X \to Y$ of symmetric T-spectra is a stable equivalence if and only if the induced map $IX_s \to IY_s$ is a level equivalence. This is exactly what is meant by a stable equivalence of symmetric T-spectra in the original sources [4], [8], [9].

Remark 11.3. The stable model structure for symmetric T-spectra given here, f-local or not, *does not* coincide with any of those appearing in the literature. In the original stable model structure for symmetric T-spectra — call it the HSS stable model structure for Hovey-Shipley-Smith [4] — a map $p : X \to Y$ is a fibration of symmetric T-spectra if and only if the underlying map Up is a fibration of T-spectra. This is not quite true here: it can be shown (in Lemma 11.11 below, subject to the assumption that $T = S^1 \wedge K$ where K is compact up to equivalence) that a map $p : X \to Y$ of symmetric T-spectra is a stable fibration if and only if it is an injective fibration *and* it restricts to a stable fibration Up of T-spectra. The HSS stable model structure is derived in Proposition 11.12.

Here are some constructions:

1) Suppose that K is a pointed simplicial presheaf and X is a symmetric T spectrum. Then $\Omega_K X = \mathbf{Hom}(K, X)$ is the symmetric T-spectrum with

$$\Omega_K X^n = \mathbf{Hom}(K, X^n).$$

The symmetric group actions and the bonding maps are defined by their adjoints.

2) Suppose that X is a symmetric T-spectrum and that $n > 0$. The symmetric spectrum $X[n]$ has $X[n]^k = X^{n+k}$, and $\alpha \in \Sigma_k$ acts on $X[n]^k$ as the element

$1 \oplus \alpha \in \Sigma_{n+k}$. The bonding map $\sigma : S^p \wedge X[n]^k \to X[n]^{p+k}$ is defined to be the composite

$$S^p \wedge X^{n+k} \xrightarrow{\sigma} X^{p+n+k} \xrightarrow{c(p,n)\oplus 1} X^{n+p+k}$$

3) The map $\tilde{\sigma} : X^n \to \Omega_T X[1]^n = \Omega_T X^{1+n}$ is the adjoint of the bonding map $T \wedge X^n \to X^{1+n}$. One shows that the diagram

$$\begin{array}{ccc} T^p \wedge X^n & \xrightarrow{T^p \wedge \tilde{\sigma}} & T^p \wedge \Omega^T X^{1+n} \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ X^{p+n} & \xrightarrow[\tilde{\sigma}]{} & \Omega_T X^{1+p+n} \end{array}$$

commutes by checking adjoints. It follows that there is a natural map

$$\tilde{\sigma} : X \to \Omega_T X[1]$$

which is induced by the adjoint bonding maps.

Remark 11.4. The map $\tilde{\sigma} : X \to \Omega_T X[1]$ has no analogue for ordinary spectra, because the functor $Y \mapsto \Omega_T Y$ in use here is real (and not fake) loops.

4) Suppose that X is a symmetric T-spectrum. Define a system $k \mapsto Q^k_\Sigma X$, $k \geq 0$ by specifying that

$$Q^k_\Sigma X = \Omega^k_T IX[k], \; k \geq 0.$$

In particular, $Q^0_\Sigma X = IX$, where $j_\Sigma : X \to IX$ is the natural choice of injective model for X. There is a natural map $Q^k_\Sigma X \to Q^{k+1}_\Sigma X$ given by the map

$$\Omega^k_T \tilde{\sigma}[k] : \Omega^k_T IX[k] \to \Omega^k_T \Omega_T IX[1][k].$$

Set $Q_\Sigma X = I(\varinjlim_k Q^k_\Sigma X)$, and write $\eta : X \to Q_\Sigma X$ for the natural composite

$$X \xrightarrow{j_\Sigma} IX = Q^0_\Sigma X \to \varinjlim_k Q^k_\Sigma X \xrightarrow{j_\Sigma} I(\varinjlim_k Q^k_\Sigma X).$$

Theorem 11.5. *Suppose that T is compact up to equivalence. Suppose that $f : X \to Y$ is a map of symmetric T-spectra such that the induced map $UX \to UY$ is a stable equivalence of T-spectra. Then f is a stable equivalence.*

Proof. There are induced natural weak equivalences

$$\begin{array}{ccc} \varinjlim_k \Omega^k_T FUX^{n+k} & \xrightarrow[\simeq]{j} & QUX^n \\ \simeq\downarrow & & \\ \varinjlim_k \Omega^k_T UIX^{n+k} & \xrightarrow[Uj_\Sigma]{\simeq} & UQ_\Sigma X^n \end{array}$$

where $j : Y \to QY$ is the natural stably fibrant model of a T-spectrum Y given by Theorem 7.4. Thus if $f : X \to Y$ induces a stable equivalence Uf, meaning a level equivalence $QUX \to QUY$, then the map of symmetric T-spectra $f_* : Q_\Sigma X \to Q_\Sigma Y$ is a level equivalence.

If a symmetric T-spectrum Z is stably fibrant then all objects $Q^k_\Sigma Z$ are stably fibrant and all maps $Z \to Q^k_\Sigma Z$ are level equivalences. It follows that $Q^k_\Sigma Z$ is stably fibrant and that the natural map $Z \to Q_\Sigma Z$ is a level equivalence.

Finally take a stably fibrant model $X \to LX$ for a symmetric T-spectrum X and consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & LX \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \simeq\ \eta} \\ Q_\Sigma X & \longrightarrow & Q_\Sigma LX \end{array}$$

The indicated maps are stable equivalences, so that X is a natural retract of $Q_\Sigma X$ in the stable homotopy category. Thus, if $f : X \to Y$ induces a stable equivalence $UX \to UY$, then the induced map $Q_\Sigma X \to Q_\Sigma Y$ is a level and hence stable equivalence, so that f is a stable equivalence. □

Overall, the meaning of Theorem 11.5 is that all of the standard properties of spectra can be bootstrapped to symmetric spectra.

Corollary 11.6. *Suppose that $T = S^1 \wedge K$ where K is compact up to equivalence, and suppose that*

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a level fibre sequence of symmetric T-spectra. Then the canonical map $X/F \to Y$ is a stable equivalence.

Proof. The induced map $U(X/F) \to UY$ is a stable equivalence of T-spectra by Lemma 8.2. □

Lemma 11.7. *Suppose that $T = S^1 \wedge K$ where K is compact up to equivalence. Suppose given a comparison of cofibre sequences*

$$\begin{array}{ccccc} A_1 & \longrightarrow & B_1 & \longrightarrow & B_1/A_1 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ A_2 & \longrightarrow & B_2 & \longrightarrow & B_2/A_2 \end{array}$$

Then if any two of f_1, f_2, f_3 are stable equivalences, then so is the third.

Proof. There is a natural isomorphism

$$\Omega_T Z[1] \cong \Omega_{S^1} \Omega_K Z[1]$$

and the canonical map $Z \to \Omega_T Z[1]$ is a level equivalence if Z is stably fibrant. It follows that the induced diagram

$$\begin{array}{ccccc} \mathbf{hom}(B_2/A_2, Z) & \longrightarrow & \mathbf{hom}(B_2, Z) & \longrightarrow & \mathbf{hom}(A_2, Z) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{hom}(B_1/A_1, Z) & \longrightarrow & \mathbf{hom}(B_1, Z) & \longrightarrow & \mathbf{hom}(A_1, Z) \end{array}$$

is a comparison of fibre sequences of infinite loop spaces for each stably fibrant object Z, and so if any two of the vertical maps is a weak equivalence then so is the third. □

Corollary 11.8. *Suppose that $T = S^1 \wedge K$ where K is compact up to equivalence. Suppose that $i : A \to B$ is a cofibration of symmetric T-spectra, and take a factorization*

$$\begin{array}{ccc} B & \xrightarrow{j} & Z \\ & {\scriptstyle \pi}\searrow & \downarrow{\scriptstyle p} \\ & & B/A \end{array}$$

such that j is a cofibration and a level equivalence and p is an injective fibration. Let F be the fibre of p. Then the induced map $A \to F$ is a stable equivalence.

Proof. It follows from Lemma 8.4 that the map $UA \to UF$ is a stable fibration. □

Corollary 11.9. *Suppose that $T = S^1 \wedge K$ where K is compact up to equivalence. Then the stable structure for symmetric T-spectra is proper.*

Proof. Suppose given a pullback diagram

$$\begin{array}{ccc} W & \xrightarrow{f_*} & X \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[f]{} & Y \end{array}$$

such that p is a level fibration with fibre F and f is a stable equivalence. Then the diagram above may be replaced up to stable equivalence by the comparison of cofibre sequences

$$\begin{array}{ccc} F & \xrightarrow{1} & F \\ \downarrow & & \downarrow \\ W & \xrightarrow{f_*} & X \\ \downarrow & & \downarrow \\ W/F & \xrightarrow[\simeq]{} & X/F \end{array}$$

by Corollary 11.6. But then f_* is a stable equivalence by Lemma 11.7. □

Lemma 11.10. *Suppose that $T = S^1 \wedge K$ where K is compact up to equivalence. Suppose that a map $p : X \to Y$ is a stable equivalence and that $Up : UX \to UY$ is a stable fibration of T-spectra. Then p is a level weak equivalence.*

Proof. The map p is a level fibration. Let F be the fibre of p and consider the fibre sequence

$$F \xrightarrow{i} X \xrightarrow{p} Y.$$

The canonical map $X/F \to Y$ is a stable equivalence of symmetric T-spectra by Corollary 11.6. The comparison of cofibre sequences

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & X/F \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & X/F & \xrightarrow[1]{} & X/F \end{array}$$

implies that the map $F \to *$ is a stable equivalence of symmetric T-spectra by Lemma 11.7. The injective model IF of F is stably fibrant and stably equivalent to $*$, so IF is levelwise contractible. The map $F \to IF$ is a levelwise equivalence, so that F is levelwise contractible.

But then $UX \to U(X/F)$ is a stable equivalence of T-spectra, so that $Up : UX \to UY$ is a stable equivalence of T-spectra as well as a stable fibration. It follows that $Up : UX \to UY$ is a level equivalence. □

Lemma 11.11. *Suppose that $T = S^1 \wedge K$, where K is compact up to equivalence. Then an injective fibration $p : X \to Y$ of symmetric T-spectra is a stable fibration if and only if $Up : UX \to UY$ is a stable fibration of T-spectra.*

Proof. If $p : X \to Y$ is a stable fibration, then p is an injective fibration, and $Up : UX \to UY$ is a stable fibration. The last claim follows from the observation that the functor V preserves stable equivalences and cofibrations.

Suppose that $i : A \to B$ is a cofibration and a stable equivalence. Then i has a factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & Z \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle q} \\ & & B \end{array}$$

where q is an injective fibration such that Uq is a stable fibration, and j is a cofibration which is a stable equivalence and has the left lifting property with respect to all such maps. In effect, there are two factorizations

$$\begin{array}{ccccc} A & \xrightarrow{j_s} & A_s & \xrightarrow{j_i} & A_{si} \\ & {\scriptstyle i}\searrow & & {\scriptstyle p_s}\searrow & \downarrow{\scriptstyle p_{si}} \\ & & & & B \end{array}$$

where p_s is a map such that Up_s is a stable fibration and j_s is a stably trivial cofibration which has the LLP with respect to all maps q such that Uq is a stable fibration. The map j_i is a level trivial cofibration and p_{si} is an injective fibration. But then Up_{si} is a strict fibration which is strictly equivalent to a stable fibration, so that Up_{si} is a stable fibration. Set $q = p_{si}$ and $j = j_i j_s$.

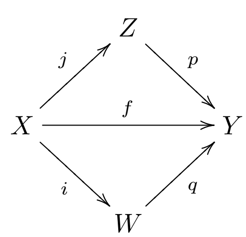
But then q is also a stable equivalence, so it is a level weak equivalence by Lemma 11.10. Thus, q is a trivial injective fibration, and therefore has the RLP with respect to all cofibrations. It follows that i is a retract of j and therefore has the LLP with respect to all injective fibrations p such that Up is a stable fibration. □

Say that $p : X \to Y$ is an HSS-fibration if $Up : UX \to UY$ is a stable fibration. Say that the map $i : A \to B$ is an HSS-cofibration if it has the left lifting property with respect to all maps which are stable equivalences and HSS-fibrations.

Lemma 11.10 implies that every map $p : X \to Y$ which is both an HSS fibration and a stable equivalence must be a level equivalence. It follows that the class of HSS cofibrations includes all maps $F_n A \to F_n B$ which are induced by the set I generators $A \to B$ for the f-local structure for pointed simplicial presheaves.

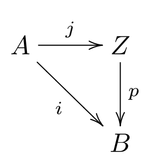
Proposition 11.12. *Suppose that $T = S^1 \wedge K$ where K is compact up to equivalence. Then the category $\mathrm{Spt}^{\Sigma}_T(\mathcal{C})$, with the classes of stable equivalences, HSS-fibrations and HSS-cofibrations as defined above, satisfies the axioms for a proper closed simplicial model category.*

Proof. The stable model structure on T-spectra is cofibrantly generated. It follows that every map $f : X \to Y$ of symmetric T-spectra has factorizations



where p is an HHS stable fibration and j is a stable equivalence which has the left lifting property with respect to all HHS fibrations, and i is an HSS cofibration and q is an HSS fibration such that Uq is a level trivial stable fibration. It follows in particular that j is an HSS cofibration and that q is a stable equivalence.

Suppose that the map $i : A \to B$ is an HSS cofibration and a stable equivalence. The i has a factorization



such that j is an HSS cofibration which has the left lifting property with respect to all HSS fibrations and is a stable equivalence, and p is an HSS fibration. Then p is a stable equivalence, and it follows that i is a retract of j. □

Lemma 11.13. *A map $p : X \to Y$ is a stable equivalence and an HSS-fibration if and only if p is a level trivial fibration.*

Proof. One direction is Lemma 11.10.

Conversely, if p is a level trivial fibration, then Up is a trivial strict fibration of T-spectra, and is therefore a stable fibration and a stable equivalence by Lemma 6.8. □

Corollary 11.14. *The cofibrations of the HHS-structure are the projective cofibrations.*

Finally, the functors U and V determine a Quillen equivalence of T-spectra with symmetric T-spectra, under suitable conditions:

Theorem 11.15. *Suppose that T is compact up to equivalence, and that the cyclic permutation $(3, 2, 1)$ acts trivially on T^3 in the f-local homotopy category. Then the functors U and V form a Quillen equivalence.*

$$V : \mathrm{Spt}_T(\mathcal{C}) \leftrightarrows \mathrm{Spt}_T^{\Sigma}(\mathcal{C}) : U.$$

The proof of Theorem 11.15 essentially coincides with that of Theorem 4.31 of [8]. The idea is to show that if X is a cofibrant T-spectrum and $j : V(X) \to LV(X)$ is a stably fibrant model of the symmetric T-spectrum X, then the composite

$$X \xrightarrow{\eta} UV(X) \xrightarrow{Uj} ULV(X) \tag{6}$$

is a stable equivalence of T-spectra. Theorem 11.15 follows from this statement, by the same formal argument as one sees in the proof of Corollary 7.9.

One shows that the composite (6) is a stable filtration with a layer filtration argument, which reduces to $X = \Sigma_T^\infty K$, In this case, η is an isomorphism, and it suffices to find a stably fibrant model $j : S_T \otimes K \to L(S_T \otimes K)$ whose underlying map of spectra is a stable equivalence.

Form the T-spectrum object $\Sigma_T^\infty(S_T \wedge K)$ in symmetric spectra, which has the object $(S_T \wedge K) \wedge T^n$ in level n. This is, alternatively, a symmetric spectrum object in T-spectra with the presheaf $T^s \wedge K \wedge T^r$ in bidegree (r, s). There is a level stable equivalence

$$T^s \wedge K \wedge T^* \to F(T^s \wedge K \wedge T^*) = F^{*,s}$$

in symmetric spectrum objects such that each object $F(T^s \wedge K \wedge T^*)$ is stably fibrant. The map

$$F(T^s \wedge K \wedge T^*) \to \Omega_T F(T^{s+1} \wedge K \wedge T^*)$$

is a stable (hence level) equivalence by Theorem 7.8. It follows from some bispectrum tricks that the map $T^* \wedge K \wedge S^0 \to F^{*,0}$ is a map of symmetric T-spectra such that the underlying map

$$U(T^* \wedge K \wedge S^0) \to U(F^{*,0})$$

is a stable equivalence of T-spectra with $U(F^{*,0})$ stably fibrant. It follows that any stably fibrant model $j : S_T \wedge K \to L(S_T \wedge K)$ induces a stable equivalence of the underlying spectra.

12 The smash product

Recall that $S_T \otimes X$ is the free symmetric spectrum associated to symmetric space X.

Lemma 12.1. *Suppose that Y is a symmetric spectrum. Then there is a canonical isomorphism of symmetric spectra*

$$Y \wedge (S_T \otimes X) \cong Y \otimes X.$$

Proof. The composite

$$Y \otimes S_T \otimes X \xrightarrow{\tau \otimes X} S_T \otimes Y \otimes X \xrightarrow{m} Y \otimes X$$

induces a natural map $Y \wedge (S_T \otimes X) \to Y \otimes X$. The unit of S_T induces a map of symmetric spaces $X \to S_T \otimes X$ which then ineduces a map of symmetric spectra $Y \otimes X \to Y \wedge (S_T \otimes X)$. These two maps are inverse to each other. □

Lemma 12.2. *There is a natural bijection*

$$\hom(X \otimes G_n(S^0), Y) \cong \hom(X, Y[n])$$

for morphisms of symmetric T-spectra.

Proof. There is a natural bijection

$$\hom(G_n S^0 \otimes X, Y) \cong \hom(X, Y[n])$$

of maps of symmetric spaces. One checks that this adjunction respects symmetric spectrum structures. □

Corollary 12.3. *There is a natural isomorphism*

$$F_n(A) \wedge F_m(B) \cong F_{n+m}(A \wedge B).$$

Proof. There are isomorphisms

$$\begin{aligned} F_n A \wedge F_m B &\cong (S_T \otimes (G_n(S^0) \wedge A)) \wedge (S_T \otimes (G_m(S^0) \wedge B)) \\ &\cong (S_T \otimes G_n(S^0) \otimes G_m(S^0)) \wedge (A \wedge B) \end{aligned}$$

There are isomorphisms of maps of symmetric spectra

$$\begin{aligned}\hom(S_T \otimes G_n(S^0) \otimes G_m(S^0), Y) &\cong \hom(S_T \otimes G_n, Y[m]) \\ &\cong \hom(S_T, Y[m+n]).\end{aligned}$$

It follows that

$$S_T \otimes G_n(S^0) \otimes G_m(S^0) \cong S_T \otimes G_{m+n}(S^0)$$

as symmetric spectra, and the desired result follows. □

Corollary 12.4. *The functor $X \mapsto X[n]$ preserves injective fibrations and trivial injective fibrations.*

Proof. The functor $Y \mapsto Y \otimes G_n(S^0)$ preserves level cofibrations and level equivalences. □

Theorem 12.5. *Suppose that $i : A \to B$ is a projective cofibration and that $j : C \to D$ is a level cofibration of symmetric T-spectra. Then the map*

$$(i, j)_* : (B \wedge C) \cup_{(A \wedge C)} (A \wedge D) \to B \wedge D$$

is a level cofibration. If i and j are both projective cofibrations then $(i, j)_*$ is a projective cofibration. If j is a stable equivalence, then $(i, j)_*$ is a stable equivalence.

Proof. For most of these statements, it suffices to assume that the projective cofibration i is a map $F_nA' \to F_nB'$ which is induced by a cofibration $i : A' \to B'$ of pointed simplicial presheaves.

If $p : X \to Y$ is an injective fibration then the induced map

$$\mathbf{Hom}(B', X)[n] \to \mathbf{Hom}(A', X)[n] \times_{\mathbf{Hom}(A',Y)[n]} \mathbf{Hom}(B', Y)[n] \qquad (7)$$

is an injective fibration which is trivial if p is trivial. This statement in the case when p is trivial implies that the map

$$(i, j)_* : (F_nB' \wedge C) \cup_{(F_nA' \wedge C)} (F_nA' \wedge D) \to F_nB' \wedge D$$

is a level cofibration.

The fact that the map (7) is an injective fibration implies that the map $(i, j)_*$ is a level weak equivalence if j is a level weak equivalence.

If $C \to D$ is a projective cofibration, then it can be approximated by cofibrations of the form $F_mC' \to F_mD'$, and then the map $(i, j)_*$ is the result of applying the functor F_{n+m} to the cofibration

$$(B' \wedge C') \cup_{(A' \wedge C')} (A' \wedge D') \to B' \wedge D'$$

of pointed simplicial presheaves.

If j is a stably trivial level cofibration, then it is in the saturation of the set S of maps generated by all α-bounded level trivial cofibrations $E \to F$ and all maps

$$F_1T \wedge F_nS^0 \to S_T \wedge F_n(S^0)$$

induced by the canonical map $F_1T \to S_T$, subject to the requirement that the map (the "tensor")

$$(C \wedge F) \cup (D \wedge E) \to D \wedge F$$

is in S for all $i : C \to D$ in S and all α-bounded cofibrations $E \to F$ of pointed simplicial sets. If the map i determines a stably trivial cofibration

$$(C \wedge F_nB) \cup (D \wedge F_nA) \to D \wedge F_nB,$$

for all cofibrations $A \to B$ of pointed simplicial presheaves, and if $E \to F$ is an α-bounded cofibration of pointed simplicial presheaves then the cofibration determined by

$$(C \wedge F) \cup (D \wedge E) \to D \wedge F$$

and the map $F_nA \to F_nB$ is the map determined by $i : C \to D$ and the map determined by applying F_n to the inclusion

$$(F \wedge A) \cup (E \wedge B) \to (F \wedge B).$$

We therefore only have to show that "tensoring" the generators of S with maps $F_nA \to F_nB$ gives stable equivalences.

The tensor of $F_nA \to F_nB$ with a map $F_1T \wedge F_mS^0 \to S_T \wedge F_mS^0$ is the same as the tensor of the map $F_1T \wedge F_{n+m}(S^0) \to S_T \wedge F_{n+m}(S^0)$ with the map $A \to B$, and this is in S. We have already seen that tensoring with $F_nA \to F_nB$ does the right thing for level trivial cofibrations. □

Lemma 12.6. *Suppose that $T = S^1 \wedge K$ where K is compact up to equivalence. Suppose that $i : A \to B$ is a projective cofibration and that $j : C \to D$ is a level cofibration of symmetric T-spectra. Then the map $(i, j)_*$ is a stable equivalence if i is a stable equivalence.*

Proof. The cofibre of $(i, j)_*$ is the smash $B/A \wedge D/C$. The quotient B/A is projective cofibrant, and so there is a level weak equivalence

$$B/A \wedge K \to B/A \wedge D/C,$$

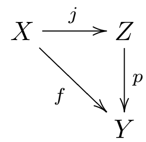
where $K \to D/C$ is a projective cofibrant model for D/C. Then the stably trivial cofibration $* \to B/A$ induces a stable equivalence

$$* \cong * \wedge K \to B/A \wedge K.$$

Thus, the cofibre $B/A \wedge D/C$ of $(i, j)_*$ is stably trivial, so that $(i, j)_*$ is a stable equivalence by Lemma 11.7. □

Corollary 12.7. *If $f : X \to Y$ is a stable equivalence and A is projective cofibrant, then the induced map $f \wedge A : X \wedge A \to Y \wedge A$ is a stable equivalence.*

Proof. Any stably trivial cofibration $j : C \to D$ induces a stable equivalence $A \wedge C \to A \wedge D$, by Theorem 12.5. The morphism $f : X \to Y$ has a factorization



such that j is a cofibration and p is a trivial injective fibration. The map p has a section $\sigma : Y \to Z$ since all symmetric T-spectra are cofibrant for the injective structure, but then σ is a stably trivial cofibration so that $A \wedge \sigma : A \wedge Y \to A \wedge X$ is a stable equivalence, and so $A \wedge p : A \wedge Z \to A \wedge Y$ is a stable equivalence. The map j is also a stably trivial cofibration, so that $A \wedge j : A \wedge X \to A \wedge Z$ is a stable equivalence. □

Corollary 12.8. *Suppose that $i : A \to B$ and $j : C \to D$ are projective cofibrations. Then the induced map*

$$(i, j)_* : (B \wedge C) \cup_{(A \wedge C)} (A \wedge D) \to B \wedge D$$

is a projective cofibration which is stably trivial if either i or j is a stable equivalence.

Remark 12.9. Take away the adjective "projective" in the statement of Corollary 12.8, and you have the description of what it means for a model structure to be monoidal, subject to having a symmetric monoidal smash product. Example: the pointed simplicial set (or presheaf) category with the obvious smash product is monoidal.

Corollary 12.10. *Suppose that $T = S^1 \wedge K$ where K is compact up to equivalence. Then the HSS-stucture on the category $\mathrm{Spt}_T^{\Sigma}(\mathcal{C})$ of symmetric T-spectra is monoidal.*

Proof. The cofibrations for the HSS-structure are the projective cofibrations. □

Here's an issue, perhaps: the HSS-structure on symmetric T-spectra is monoidal when it exists, which is so far only in the case where T is a suspension of an object which is compact up to equivalence. On the other hand, the stable structure for symmetric T-spectra and Corollary 12.8 both obtain in extreme generality, and there is a universal description of a derived tensor project: set

$$X \wedge_{\Sigma} Y = X' \wedge Y'$$

where $\pi_X : X' \to X$ and $\pi_Y : Y' \to Y$ are projective cofibrant models for X and Y respectively. The standard description of a monoidal model structure may not be exactly the right thing.

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