**Lecture 01** (December 22, 2015)

## 1 Modules and exactness

Suppose that $R$ is an associative ring with 1.

In most commutative cases, $R$ is either the integers $\mathbb{Z}$ or some field $k$.

**Example**: Suppose that $k$ is a field and $G$ is a group. The *group-algebra* $k(G)$ over $k$ is the direct sum

$$k(G) = \bigoplus_{g \in G} k,$$

with elements written as finite sums $\sum_{g \in G} \lambda_g \cdot g$, with $\lambda_g \in k$ and all but finitely many $\lambda_g = 0$. The "rule"

$$(\lambda_g \cdot g)(\lambda_h \cdot h) = (\lambda_g \lambda_h) \cdot (gh)$$

defines the algebra structure on $k(G)$, with multiplicative identity $1 = 1 \cdot e$, where $e$ is the identity element of $G$.

A $k(G)$-module $M$ is a $k$-vector space $M$, with bilinear map

$$* : k(G) \times M \to M$$

with $(r, m) \mapsto r*m$, such that $r*(s*m) = (r \cdot s)* m$ and $1*m = m$, or equivalently $M$ is a $k$-vector space equipped with a group homomorphism

$$G \to \mathrm{Aut}_k(M).$$

$k(G)$-modules are often called $G$-modules for that reason.

Not even *that* is the most enlightened way to describe a $k(G)$-module. A group $G$ can be thought of as a category (actually a groupoid) with one object $*$ and a morphism $* \xrightarrow{g} *$ for every $g \in G$. Then a $k(G)$-module is a functor $M : G \to k - \mathbf{Mod}$ which takes values in the category of $k$-vector spaces.

**NB**: I've only based these notions on fields $k$ and their vector spaces to make them seem real. The object $k$ could be a ring; then $k(G)$ is a $k$-algebra still and a $k(G)$-module is a $k$-module $M$ equipped with a group homomorphism $G \to \mathrm{Aut}_k(M)$.

Now we recall some basic definitions and facts about $R$-modules.

Suppose that $f : M \to N$ is an $R$-module homomorphism. Then the *kernel* $\ker(f)$ of $f$ is defined by

$$\ker(f) = \{\text{all } x \in M \text{ such that } f(x) = 0\}.$$

$\ker(f)$ is plainly a submodule of $M$. The *image* $\operatorname{im}(f)$ of $f$ is the submodule of $N$ consisting of all $y \in N$ such that $y = f(x)$ for some $x \in M$. The *cokernel* of $f$ $\operatorname{cok}(f)$ is defined to be the quotient

$$\operatorname{cok}(f) = N/\operatorname{im}(f).$$

A sequence

$$M \xrightarrow{f} M' \xrightarrow{g} M''$$

of $R$-module homomorphisms is said to be *exact* if $\ker(g) = \operatorname{im}(f)$. Equivalently, the sequence is exact if $g \cdot f = 0$ and for all $y \in M'$ with $g(y) = 0$ there is an $x \in M$ such that $f(x) = y$.

A sequence

$$M_1 \to M_2 \to \cdots \to M_n$$

of $R$-module homomorphisms is said to be exact if $\ker = \operatorname{im}$ everywhere.

**Example 1.1.** The sequence

$$0 \to \ker(f) \to M \xrightarrow{f} N \to \operatorname{cok}(f) \to 0$$

is exact for all $R$-module homomorphisms $f$.

Note that

$$0 \to M \xrightarrow{f} N$$

is exact if and only if $f$ is a monomorphism (injective), and that

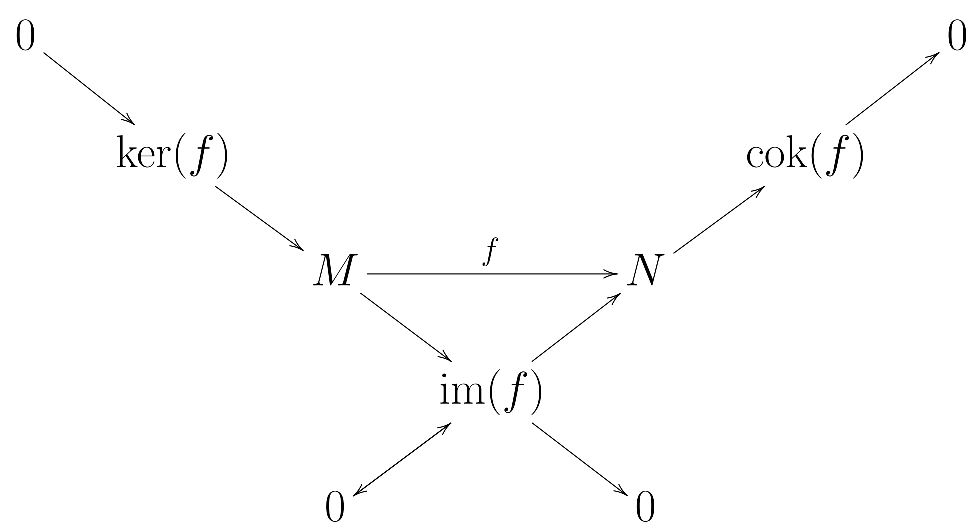
$$M \xrightarrow{f} N \to 0$$

is exact if and only if $f$ is an epimorphism (surjective).

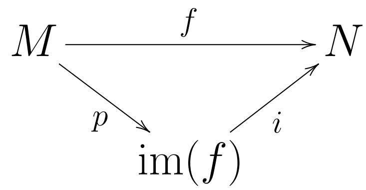
A *short exact sequence* always means an exact sequence of the form

$$0 \to M \to M' \to M'' \to 0.$$

All of these definitions for a map $f : M \to N$ can be summarized in the following diagram of exact sequences

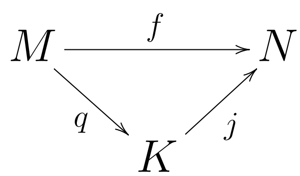


**Exercise**: The diagram

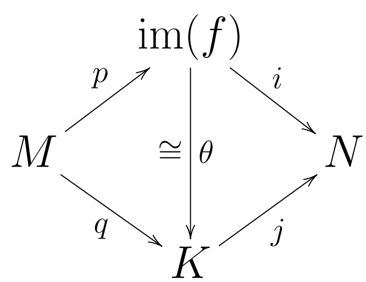



is usually called an *epi-monic factorization* of $f : M \to N$, meaning that $p$ is surjective, $i$ is injective and $f = i \cdot p$.

Show that if there is another factorization




of $f$ with $q$ an epimorphism and $j$ a monomorphism, then there is a unique $R$-module isomorphism $\theta : \mathrm{im}(f) \xrightarrow{\cong} K$ which makes the diagram




commute.

Here are some fundamental exactness results:

**Lemma 1.2** (**Snake Lemma**). *Suppose given a diagram of $R$-module homomorphisms*

$$\begin{array}{ccccccc} & & A_1 & \longrightarrow & A_2 & \xrightarrow{p} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow[i]{} & B_2 & \longrightarrow & B_3 & & \end{array}$$

*Then there is an induced exact sequence*

$$\ker(f_1) \to \ker(f_2) \to \ker(f_3) \xrightarrow{\partial} \operatorname{cok}(f_1) \to \operatorname{cok}(f_2) \to \operatorname{cok}(f_3).$$

*Proof.* The boundary homomorphism $\partial$ is defined by $\partial(y) = [z]$ for $y \in \ker(f_3)$, where $y = p(x)$, and $f_2(x) = i(z)$.

To see that $\partial$ is well defined, if $y = p(x')$ (and $f_2(x') = i(z')$), then $p(x-x') = 0$ so $x - x' = i(v)$ for some $v \in A_1$. Then $z - z' = f_1(v)$ in $B_1$ so that $[z] = [z']$ in $\operatorname{cok}(f_1)$.

Let's show that the sequence

$$\ker(f_2) \xrightarrow{p_*} \ker(f_3) \xrightarrow{\partial} \operatorname{cok}(f_1)$$

is exact.

First of all, if $y = p(x)$ for some $x \in \ker(f_2)$, then $f_2(x) = 0$ and $\partial(y) = 0$ by definition.

If $\partial(y) = 0$ then $z = f_1(w)$ for some $w \in A_1$, so that $p(x - i(w)) = p(x) = y$ while $f_2(x - i(w)) =$

0, so that $y$ is in the image of $p_* : \ker(f_2) \to \ker(f_3)$.

The other three exactness statements have similar proofs. □

**Remark**: The proof of the Snake Lemma is a classic example of an "element chasing" argument. Many proofs in classical homological algebra have this flavour, and it's the method of last resort.

**Lemma 1.3** ((3 × 3)**-Lemma).** *Suppose given a commutative diagram of $R$-module homomorphisms*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & A_1 & \rightarrow & A_2 & \rightarrow & A_3 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & B_1 & \xrightarrow{i} & B_2 & \xrightarrow{p} & B_3 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & C_1 & \rightarrow & C_2 & \rightarrow & C_3 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

*in which all columns are exact. Then*

1) *if the $A$ and $B$ rows are exact, then the $C$ row is exact,*

*2) if the $C$ and $C$ rows are exact, then the $A$ row is exact,*

*3) if the $A$ and $C$ rows are exact and if $p \cdot i = 0$ in the $B$ row, then the $B$ row is exact.*

*Proof.* The Snake Lemma implies directly that

- if the $A$-row and $B$-row are both exact then the $C$-row is exact,
- if the $B$-row and $C$-row are both exact then the $A$-row is exact.

Thus, suppose that the $A$-row and $C$-row are both exact.

The Snake Lemma applied to the vertical columns implies that $i$ is a monomorphism and $p$ is an epimorphism. Thus, it suffices to show that $\mathrm{im}(i) = \ker(p)$.
Since $p \cdot i = 0$, there is a comparison of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_3 & \longrightarrow & \mathrm{cok}(i) & \longrightarrow & C_3 & \longrightarrow & 0 \\ & & {\scriptstyle 1}\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & A_3 & \longrightarrow & B_3 & \longrightarrow & C_3 & \longrightarrow & 0 \end{array}$$

and it follows that the induced map $\mathrm{cok}(i) \to B_3$ is an isomorphism. $\square$

**Lemma 1.4** (**5-Lemma**)**.** *Suppose given a commutative diagram of $R$-module homomorphisms*

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \xrightarrow{g_1} & A_5 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \downarrow{\scriptstyle h_4} & & \downarrow{\scriptstyle h_5} \\ B_1 & \xrightarrow[f_2]{} & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \xrightarrow[g_2]{} & B_5 \end{array}$$

*with exact rows and such that $h_1, h_2, h_4, h_5$ are isomorphisms. Then $h_3$ is an isomorphism.*

*Proof.* Use the Snake Lemma and the observation that there is an induced diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{cok}(f_1) & \longrightarrow & A_3 & \longrightarrow & \ker(g_1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle h_3} & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & \mathrm{cok}(f_2) & \longrightarrow & B_3 & \longrightarrow & \ker(g_2) & \longrightarrow & 0 \end{array}$$

□

## 2 Chain complexes

**Definition 2.1.** A *chain complex* $C$ in $R$-modules is a sequence of $R$-module homomorphisms

$$\ldots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} C_{-1} \xrightarrow{\partial} \ldots$$

such that $\partial^2 = 0$ (or that $\mathrm{im}(\partial) \subset \ker(\partial)$) everywhere. In this case $C_n$ is often called the module of $n$-chains.

A *morphism* $f : C \to D$ of chain complexes consists of $R$-module maps $f_n : C_n \to D_n$, $n \in \mathbb{Z}$ such that all diagrams

$$\begin{array}{ccc} C_n & \xrightarrow{f_n} & D_n \\ \partial \downarrow & & \downarrow \partial \\ C_{n-1} & \xrightarrow[f_{n-1}]{} & D_{n-1} \end{array}$$

commute.

The chain complexes and their morphisms form a category, denoted by $Ch(R)$.

**Remark 2.2.**

- If $C$ is a chain complex such that $C_n = 0$ for $n < 0$, then $C$ is said to be an *ordinary* chain complex. For such a thing, we usually neglect the copies of 0, and write

  $$\to C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

  for the chain complex. The full subcategory of $Ch(R)$ whose objects are the ordinary chain complexes is denoted by $Ch_+(R)$.

- Chain complexes indexed by the integers as described above are often called *unbounded* complexes. You are to think that ordinary chain complexes are analogous to spaces, while unbounded complexes are analogous to spectra.

- Chain complexes with their non-trivial terms concentrated in negative degrees, namely those of the form

$$\cdots \to 0 \to C_0 \to C_{-1} \to \dots$$

are usually called *cochain complexes* in the literature, and are often written (classically) as

$$C^0 \to C^1 \to C^2 \to \dots .$$

Both notations are in common use, and a continuing need for translation between them afflicts us all.

Morphisms of chain complexes have kernels and cokernels, defined degreewise, and a sequence of chain complex homomorphisms

$$C \to D \to E$$

is exact if and only if all sequences of $R$-module homomorphisms

$$C_n \to D_n \to E_n$$

are exact.

Given a chain complex $C$ :

$$\cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \dots$$

write

$$Z_n = Z_n(C) = \ker(\partial : C_n \to C_{n-1}), \text{ and}$$
$$B_n = B_n(C) = \mathrm{im}(\partial : C_{n+1} \to C_n).$$

Here $Z_n$ is called the module of *$n$-cycles*, and $B_n$ is the group of *$n$-boundaries* of $C$. Then $\partial^2 = 0$ so that $B_n \subset Z_n$ and the $n^{th}$ *homology group* $H_n(C)$ of $C$ is defined by

$$H_n(C) = Z_n(C)/B_n(C).$$

Any chain map $f : C \to D$ induces $R$-module homomorphisms

$$f_* : H_n(C) \to H_n(D),\ n \in \mathbb{Z}.$$

**Definition 2.3.** The chain map $f : C \to D$ is said to be a *homology isomorphism* (equivalently *quasi-isomorphism*, *acyclic map*, or *weak equivalence*) if all induced maps $f_* : H_n(C) \to H_n(D),\ n \in \mathbb{Z}$ are isomorphisms.

One often says that a complex $C$ is *acyclic* if the map $0 \to C$ is a homology isomorphism, or equivalently if $H_n(C) \cong 0$ for all $n$.

**Lemma 2.4.** *Any short exact sequence*

$$0 \to C \to D \to E \to 0$$

*induces a natural long exact sequence*

$$\dots \xrightarrow{\partial} H_n(C) \to H_n(D) \to H_n(E) \xrightarrow{\partial} H_{n-1}(C) \to \dots$$

*in homology modules.*

*Proof.* The short exact sequence induces comparisons of exact sequences

$$\begin{array}{ccccccc} & C_n/B_n(C) & \longrightarrow & D_n/B_n(D) & \longrightarrow & E_n/B_n(E) & \longrightarrow 0 \\ & \downarrow \partial_* & & \downarrow \partial_* & & \downarrow \partial_* & \\ 0 \longrightarrow & Z_{n-1}(C) & \longrightarrow & Z_{n-1}(D) & \longrightarrow & Z_{n-1}(E) & \end{array}$$

There is a natural exact sequence

$$0 \to H_n(C) \to C_n/B_n(C) \xrightarrow{\partial_*} Z_{n-1}(C) \to H_{n-1}(C) \to 0$$

for each chain complex $C$. Now use the Snake Lemma. $\square$

Here's a sample application:

**Corollary 2.5.** *Supppose given a comparison*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E & \longrightarrow & 0 \\ & & f_C \downarrow & & \downarrow f_D & & \downarrow f_E & & \\ 0 & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' & \longrightarrow & 0 \end{array}$$

*of exact sequences of chain complexes. If any two of the maps $f_C$, $f_D$ and $f_E$ are homology isomorphisms, then so is the third.*

*Proof.* Let's suppose that $f_C$ and $f_E$ are homology isomorphisms. The comparison of short exact sequences induces a comparison of exact sequences

$$\begin{array}{ccccccccc}
H_{n+1}(E) & \xrightarrow{\partial} & H_n(C) & \longrightarrow & H_n(D) & \longrightarrow & H_n(E) & \xrightarrow{\partial} & H_{n-1}(C) \\
\cong \downarrow & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong \\
H_{n+1}(E') & \xrightarrow[\partial]{} & H_n(C') & \longrightarrow & H_n(D') & \longrightarrow & H_n(E') & \xrightarrow[\partial]{} & H_{n-1}(C')
\end{array}$$

Use the 5-lemma to show that the induced map $H_n(D) \to H_n(D')$ is an isomorphism. $\square$

## 3 Ordinary chain complexes, simplicial sets

### Singular homology

Write $|\Delta^n|$ for the subspace of $\mathbb{R}^{n+1}$ which is defined by

$$|\Delta^n| = \{(t_0, t_1, \dots, t_n) \mid t_i \geq 0, \sum_{i=0}^{n} t_i = 1\}.$$

Then

- $|\Delta^1|$ is the one-point space $\{1\} \subset \mathbb{R}$,
- $|\Delta^1|$ is the line segment joining the points $(0,1)$ and $(1,0)$ in the plane $\mathbb{R}^2$,
- $|\Delta^2|$ is the triangle with vertices $(0,0,1)$, $(0,1,0)$ and $(0,0,1)$ in $\mathbb{R}^3$,

- $|\Delta^3|$ is a tetrahedron in real 4-space,

and so on. $|\Delta^n|$ is called the *topological standard $n$-simplex*.

There are special continuous maps

$$d^i : |\Delta^{n-1}| \to |\Delta^n|, \quad 0 \leq i \leq n,$$

where

$$d^i(t_0, t_1, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

is defined by putting 0 in the $i^{th}$ place and shuffling the other entries accordingly. One can check (exercise) that there are relations

$$d^j d^i = d^i d^{j-1} : |\Delta^{n-2}| \to |\Delta^n| \qquad (1)$$

for $i < j$.

Suppose that $X$ is a topological space, and write

$$S(X)_n = \{|\Delta^n| \to X\}$$

for the set of all continuous maps $|\Delta^n| \to X$, which maps are called the *$n$-simplices* of $X$. The *coface maps $d^i$* induce functions

$$d_i : S(X)_{n+1} \to S(X)_n, \ 0 \leq i \leq b,$$

which are defined, respectively, by precomposition with $d^i$: if $\sigma : |\Delta^n| \to X$ is an $n$-simplex of $X$, then the $i^{th}$ *face* $d_i(\sigma)$ is the $(n-1)$-simplex which is defined by the composite

$$|\Delta^{n-1}| \xrightarrow{d^i} |\Delta^n| \xrightarrow{\sigma} X.$$

The the "cosimplicial identities" (1) induce the "simplicial identities"

$$d_i d_j = d_{j-1} d_i : S(X)_n \to S(X)_{n-2} \tag{2}$$

for $i < j$.

Write

$$\mathbb{Z}S(X)_n = \bigoplus_{\sigma:|\Delta^n|\to X} \mathbb{Z}$$

for the free abelian group on the set $S(X)_n$ of $n$-simplices of $X$. The face maps $d_i : S(X)_n \to S(X)_{n-1}$ extend to unique abelian group homomorphisms

$$d_i : \mathbb{Z}S(X)_n \to \mathbb{Z}S(X)_{n-1}$$

with

$$d_i(\sum_\sigma n_\sigma \cdot \sigma) = \sum_\sigma n_\sigma \cdot d_i\sigma.$$

By the uniqueness of the extensions, these abelian group homomorphisms also satsify the simplicial identites (2).

Write:

$$\partial = \sum_{i=0}^{n} (-1)^i d_i : \mathbb{Z}S(X)_n \to \mathbb{Z}S(X)_{n-1}$$

for the alternating sum of the abelian group face maps $d_i$. It is a basic consequence (exercise: everybody must do this) that the simplicial identities (2) imply that the composite abelian group homomorphism

$$\mathbb{Z}S(X)_n \xrightarrow{\partial} \mathbb{Z}S(X)_{n-1} \xrightarrow{\partial} \mathbb{Z}S(X)_{n-2}$$

is the 0 homomorphism.

The resulting complex $\mathbb{Z}S(X)_n, \partial$, $n \geq 0$, is called the *singular complex* of the space $X$. There are various notations, namely

$$\mathbb{Z}S(X) = C_*(X, \mathbb{Z}).$$

Its homology groups $H_n(X, \mathbb{Z})$, $n \geq 0$, are called the *integral singular homology groups* of $X$.

2 comments:

1) The construction is functorial: any continuous map $f : X \to Y$ induces functions $f_* : S_n(X) \to S_n(Y)$ by composition with $f$. These functions re-

spect the face maps, in the sense that the diagrams

$$\begin{array}{ccc} S_n(X) & \xrightarrow{f_*} & S_n(Y) \\ {\scriptstyle d_i}\downarrow & & \downarrow{\scriptstyle d_i} \\ S_{n-1}(X) & \xrightarrow[f_*]{} & S_{n-1}(Y) \end{array}$$

commute. It follows that $f : X \to Y$ induces a chain map

$$f_* : \mathbb{Z}S(X) \to \mathbb{Z}S(Y),$$

and therefore induces homomorphisms

$$f_* : H_n(X, \mathbb{Z}) \to H_n(Y, \mathbb{Z}),\ n \geq 0.$$

2) Suppose that $A$ is an abelian group. Then tensoring with $A$ gives a chain complex

$$\mathbb{Z}S(X) \otimes A =: C_*(X, A),$$

which is again functorial in spaces $X$. The corresponding homology groups

$$H_n(X, A) := H_n(\mathbb{Z}S(X) \otimes A),\ n \geq 0,$$

are the *singular homology groups of $X$ with coefficients in $A$*. These chain complex and homology constructions are again functorial in spaces $X$, and are also functorial in abelian groups $A$ (exercise).

If $R$ is an associative ring with unit, then the chain complex $\mathbb{Z}S(X) \otimes R$ and the homology groups

$$H_n(X, R) := H_n(\mathbb{Z}S(X) \otimes R),\ n \geq 0,$$

are defined according to the prescription above. Alternatively, one can take $RS(X)_n$ to be the free $R$-module on the set $S(X)_n$ of simplices, for all $n \geq 0$, and then

$$H_n(X, R) = H_n(RS(X)),\ n \geq 0,$$

are the homology groups of the resulting complex $RS(X)$.

**Classifying spaces**

Write

$$\mathbf{n} = \{0, 1, \dots, n\}$$

for the collection of counting numbers from 0 to $n$, with their natural ordering. This set $\mathbf{n}$ is a partially ordered set, but it can also be identified with a tiny category whose objects are the underlying set, and with a unique morphism $i \to j$ if and only if $i \leq j$ in the ordering. The collection of all such *finite ordinal numbers* $\mathbf{n}$, $n \geq 0$, with the collection of all order-preserving functions (aka. functors) between them is called the *ordinal number category*, denoted by $\mathbf{\Delta}$.

Suppose that $C$ is some category. Then a functor $\alpha : \mathbf{n} \to C$ is completely determined by the objects $\alpha(i) \in C$ and the images

$$\alpha(i) \to \alpha(i+1)$$

of the morphisms $i \leq i+1$ of $\mathbf{n}$. Every other relation $i \leq j$ must be sent to the composition of the string

$$\alpha(i) \to \alpha(i+1) \to \cdots \to \alpha(j-1) \to \alpha(j)$$

since $\alpha$ is a functor. It follows that the functors $\mathbf{n} \to C$ can be identified with composable strings of morphisms

$$a_0 \to a_1 \to \cdots \to a_n$$

of length $n$ in $C$.

An ordinal number map $\mathbf{n-1} \to \mathbf{n}$ which is a monomorphism as order-preserving functions misses exactly one element $i \in \mathbf{n}$. We call this morphism $d^i$ and identify it with the string

$$0 \to 1 \to \cdots \to i-1 \to i+1 \to \cdots \to n$$

of length $n-1$ in $\mathbf{n}$. There is one of these for every $0 \leq i \leq n$, and this is a characterization of all ordinal number monomorphisms $\mathbf{n-1} \to \mathbf{n}$. These maps are called *cofaces*.

An ordinal number map $\mathbf{n+1} \to \mathbf{n}$ which is surjective as a function takes two numbers $j$ and $j+1$ to $j \in \mathbf{n}$ and shuffles the other numbers accordingly. This map is called $s^j$ and is identified with the string

$$0 \to \cdots \to j \to j \to j+1 \to \cdots \to n$$

of length $n+1$ in $\mathbf{n}$. There is one of these maps for every $0 \leq j \leq n$, and this is a characterization of all ordinal number surjections $\mathbf{n}+\mathbf{1} \to \mathbf{n}$. These maps are called *codegeneracies*.

**Remark 3.1.** The cofaces $d^i$ and the codegeneracies $s^j$ together satisfy the following list of relations

$$\begin{aligned} &d^j d^i = d^i d^{j-1} \quad \text{if } i < j, \\ &s^j d^i = \begin{cases} d^i s^{j-1} & \text{if } i < j, \\ 1 & \text{if } i = j, j+1, \\ d^{i-1} s^j & \text{if } i > j+1, \end{cases} \\ &s^j s^i = s^i s^{j+1} \quad \text{if } i \leq j. \end{aligned} \tag{3}$$

The maps $d^i$, $s^j$ (in all degrees), together with the *cosimplicial identities* above give a generators and relations description of the ordinal number category $\mathbf{\Delta}$ [1, p.178]. This means that a functor $\mathbf{\Delta} \to C$ is defined by definitions of the images of the maps $d^i$ and $s^j$, so long as those images satisfy the cosimplicial identities (3).

Suppose that $C$ is a category which is *small* in the sense that its morphisms form a set. Then the functors $\mathbf{n} \to C$ also form a set, which is denoted $BC_n$. Precomposing with any ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ defines a function

$$\theta^* : BC_n \to BC_m$$

which sends a functor (string of arrows of length $n$) $\sigma : \mathbf{n} \to C$ to the composite

$$\mathbf{m} \xrightarrow{\theta} \mathbf{n} \xrightarrow{\sigma} C.$$

In other words, the assignment

$$\mathbf{n} \mapsto BC_n$$

defines a contravariant set-valued functor (ie. morphism reversing functor)

$$BC : \mathbf{\Delta}^{op} \to \mathbf{Set}$$

which is called the *classifying space* or *nerve* of the category $C$.

The nerve $BC$ of a category $C$ is a simplicial set. In general, a *simplicial set* $X$ is a contravariant functor $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$ which is defined on the ordinal number category, and takes values in sets. Morphisms of simplicial sets are natural transformations: the resulting category is called the *category of simplicial sets* and is denoted by $s\mathbf{Set}$.

One usually writes

$$X_n = X(\mathbf{n})$$

for a simplicial set $X$. The elements of the set $X_n$ are called the *$n$-simplices* of $X$. One also writes

$$d_i = d^{i*} : X_n \to X_{n-1}$$

and calls them the *face maps* of $X$. The maps

$$s_j = s^{j*} : X_n \to X_{n+1}$$

are called the *degeneracies* or *degeneracy maps* of $X$.

The assignment $C \mapsto BC$ defines a functor on small categories which takes values in the category $s\mathbf{Set}$ of simplicial sets.

Here's an observation. Suppose $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number morphism. Then $\theta$ induces a continuous map

$$\theta_* : |\Delta^m| \to |\Delta^n|,$$

where

$$\theta_*(s_0, \dots, s_m) = (t_0, \dots, t_n),$$

and

$$t_j = \sum_{\theta(i)=j} s_i.$$

In particular, if $\theta^{-1}(j) = \emptyset$, then $t_j = 0$.

**Example/exercise**: Show that the map $d^i_* : |\Delta^{n-1}| \to |\Delta^n|$ which is induced by $d^i : \mathbf{n-1} \to \mathbf{n}$ is the map

$$(s_0, \dots, s_{n-1}) \mapsto (s_0, \dots, s_{i-1}, 0, s_i, \dots, s_{n-1}),$$

aka. the map $d^i$ which was discussed above.

The singular complex $S(X)$ for a space $X$ as discussed above is a piece of a simplicial set with

$$S(X)_n = \{\ |\Delta^n| \to X\ \} \text{ (continuous maps).}$$

Precomposing with the maps $\theta_* : |\Delta^m| \to |\Delta^n|$ gives the simplicial structure for the singular complex $S(X)$, *exactly as for the nerve of a category* $BC$.

Now here's a general construction: every simplicial set $Y$ has an associated simplicial abelian group $\mathbb{Z}(Y)$ defined by composing $Y : \mathbf{\Delta}^{op} \to \mathbf{Set}$ with the free abelian group functor $\mathbb{Z} : \mathbf{Set} \to \mathbf{Ab}$. There is a chain complex $\mathbb{Z}(Y)$ with $n$-chains

$$\mathbb{Z}(Y)_n = \mathbb{Z}(Y(\mathbf{n})),$$

and with boundary

$$\partial = \sum_{i=0}^{n} (d^{i*}) = \sum_{i=0}^{n} d_i : \mathbb{Z}(Y)_n \to \mathbb{Z}(Y)_{n-1}.$$

The singular homology groups $H_n(Y,\mathbb{Z})$ for the simplicial set $Y$ are defined by

$$H_n(Y,\mathbb{Z}) = H_n\mathbb{Z}S(Y),\ n \geq 0.$$

Similarly, if $A$ is an abelian group, then the homology groups $H_n(Y,A)$ *with coefficients in* $A$ are defined by

$$H_n(Y,A) = H_n(\mathbb{Z}S(Y) \otimes A),\ n \geq 0.$$

**Example**: Suppose that $G$ is a group, and identify $G$ with a category having one object. The classifying space or nerve of $G$ is the simplicial set $BG$, and the homology groups

$$H_n(BG,\mathbb{Z}),\ H_n(BG,A),\ n \geq 0,$$

are homology groups of the group $G$ (with trivial co-efficients, meaning no group action).

## References


[1] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.