Lecture 003 (September 30, 2009)

6 Example: Chain homotopy

Suppose that C is an ordinary chain complex. Let C^I be the complex with

$$C_n^I = C_n \oplus C_n \oplus C_{n+1}$$

for $n > 0$, and with

$$C_0^I = \{(x, y, z) \in C_0 \oplus C_0 \oplus C_1 \mid (x-y) + \partial(z) = 0\}.$$

The boundary map $\partial : C_n^I \to C_{n-1}^I$ is defined by

$$\partial(x, y, z) = (\partial(x), \partial(y), (-1)^n(x - y) + \partial(z)).$$

Here's another construction: $\tilde{C}$ is the chain complex with

$$\tilde{C}_n = C_n \oplus C_{n+1}$$

for $n > 0$ and

$$\tilde{C}_0 = \{(x, z) \in C_0 \oplus C_1 \mid x + \partial(z) = 0\}.$$

The boundary $\partial : \tilde{C}_n \to \tilde{C}_{n-1}$ of $\tilde{C}$ is defined by

$$\partial(x, z) = (\partial(x), (-1)^n x + \partial(z)).$$

Lemma 6.1. *The complex $\tilde{C}$ is acyclic.*

Proof. If $\partial(x, z) = 0$ then $\partial(x) = 0$ and $\partial(z) = (-1)^{n+1}x$. It follows that

$$\partial((-1)^{n+1}z, 0) = (x, z)$$

if (x, z) is a cycle, so that (x, z) is a boundary. □

There is a pullback diagram of chain complex maps

$$\begin{array}{ccc} C^I & \xrightarrow{\alpha} & \tilde{C} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ C \oplus C & \xrightarrow[\beta]{} & C \end{array}$$

in which the vertical maps p and p' are projections defined in each degree by $p(x, y, z) = (x, y)$ and $p'(x, z) = x$, respectively. The map α is defined by $\alpha(x, y, z) = (x - y, z)$, while $\beta(x, y) = x - y$.

The map p' is a fibration, and fibrations are closed under pullback, so p is also a fibration. The chain maps α and β are surjective in all degrees, and the diagram above expands to a comparison of short exact sequences

$$\begin{array}{ccccccccc} & & & \overset{s}{\nearrow} & C^I & \xrightarrow{\alpha} & \tilde{C} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} & & \\ 0 & \longrightarrow & C & \xrightarrow[\Delta]{} & C \oplus C & \xrightarrow[\beta]{} & C & \longrightarrow & 0 \end{array}$$

where Δ is the diagonal map. Lemma 6.1 and a long exact sequence argument together imply that the map s is a weak equivalence.

We have therefore constructed a functorial diagram

$$\begin{array}{ccc} & & C^I \\ & \overset{s}{\nearrow} & \downarrow p \\ C & \xrightarrow[\Delta]{} & C \oplus C \end{array} \tag{1}$$

in which p is a fibration and s is a weak equivalence. In the terminology around closed model categories, this is called a path object.

In that same language, one also says that any commutative diagram of chain maps

$$\begin{array}{ccc} & & C^I \\ & \overset{h}{\nearrow} & \downarrow p \\ D & \xrightarrow[(f,g)]{} & C \oplus C \end{array} \tag{2}$$

is a right homotopy between the chain maps $f, g : D \to C$

The map h, if it exists, is defined by

$$h(x) = (f(x), g(x), s(x))$$

for a collection of R-module maps $s : D_n \to C_{n+1}$, and the fact that h is a chain map forces

$$s(\partial(x)) = (-1)^n(f(x) - g(x)) + \partial(s(x))$$

for $x \in D_n$. Thus

$$(-1)^n s(\partial(x)) = (f(x) - g(x)) + \partial((-1)^n s(x)),$$

so that

$$(-1)^n s(\partial(x)) + \partial((-1)^{n+1} s(x) = f(x) - g(x).$$

Thus the maps $x \mapsto (-1)^{n+1} s(x)$, $x \in D_n$ associated to the right homotopy h define a chain homotopy between the chain maps f and g. Further, all chain homotopies arise in this way.

Exercise: Show that there is a functorial diagram of the form (1) for unbounded chain complexes C, such that the corresponding right homotopies (2) define chain homotopies between maps $f, g : D \to C$ of unbounded chain complexes.

7 Homotopical algebra

Suppose, thoughout this section, that $\mathcal{M}$ is a closed model category.

Here's the meaning of the word "closed":

Lemma 7.1. *1) A map $i : A \to B$ is a cofibration if and only if it has the left lifting property with respect to all trivial fibrations.*

2) The map i is a trivial cofibration if and only if it has the left lifting property with respect to all fibrations.

3) A map $p : X \to Y$ is a fibration if and only if it has the right lifting property with respect to all trivial cofibrations.

4) The map p is a trivial fibration if and only if it has the right lifting property with respect to all cofibrations.

Proof. I'll prove statement 2). The rest are similar.

First of all, if i is a trivial cofibration, then it has the left lifting property with respect to all fibrations by **CM4**.

Suppose that i has the advertised lifting property. The map i has a factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & X \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle p} \\ & & B \end{array}$$

where j is a trivial cofibration and p is a fibration. By the assumption on i the lifting exists in the diagram

$$\begin{array}{ccc} A & \xrightarrow{j} & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow[1]{} & B \end{array}$$

It follows that i is a retract of j and is therefore a trivial cofibration by **CM4**. $\square$

Corollary 7.2. *1) The classes of cofibrations and trivial cofibrations are closed under compositions and pushout. Any isomorphism is a trivial cofibration.*

2) The classes of fibrations and trivial fibrations are closed under composition and pullback. Any isomorphism is a trivial fibration.

Remark 7.3. Another practical consequence of Lemma 7.1 is that, in order to describe a closed model structure, one needs only specify the weak equivalences and either the cofibrations or fibrations. We saw this in the descriptions of the model structures for the chain complex categories and for spaces.

Now let's discuss the various notions of homotopy that one has in a model structure.

Definition 7.4. 1) A *path object* for an object Y of $\mathcal{M}$ is a commutative diagram

$$\begin{array}{ccc} & & Y^I \\ & \overset{s}{\nearrow} & \downarrow p \\ Y & \xrightarrow[\Delta]{} & Y \times Y \end{array}$$

such that Δ is the diagonal map, s is a weak equivalence and p is a fibration.

2) A *right homotopy* between maps $f, g : X \to Y$ is a commutative diagram

$$\begin{array}{ccc} & & Y^I \\ & \overset{h}{\nearrow} & \downarrow p \\ X & \xrightarrow[(f,g)]{} & Y \times Y \end{array}$$

where p is the fibration appearing in some path object for Y. Say that f is *right homotopic* to g if such a right homotopy exists and write $f \sim_r g$.

Example 7.5. 1) Path objects abound in nature, since the diagonal map $\Delta : Y \to Y \times Y$ factorizes as a fibration following a trivial cofibration, by **CM5**.

2) Chain homotopy is a type of right homotopy in both $Ch_+(R)$ and $Ch(R)$.

Here's the dual definition:

Definition 7.6. 1) A *cylinder object* for an object $X \in \mathcal{M}$ is a commutative diagram

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\nabla} & X \\ i \downarrow & \underset{\sigma}{\nearrow} & \\ X \otimes I & & \end{array}$$

where ∇ is the "fold" map, i is a cofibration and σ is a weak equivalence.

2) A *left homotopy* between maps $f, g : X \to Y$ is a commutative diagram

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{(f,g)} & Y \\ {\scriptstyle i}\downarrow & \nearrow_{h} & \\ X \otimes I & & \end{array}$$

where the map i is the cofibration appearing in some cylinder object for X. Say that f is *left homotopic* to g if such a left homotopy exists, and write $f \sim_l g$.

Example 7.7. 1) Suppose that X is a CW-complex and I is the unit interval. Then the standard picture

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\nabla} & X \\ {\scriptstyle i}\downarrow & \nearrow_{pr} & \\ X \times I & & \end{array}$$

is a cylinder object for X. Note that $X \times I$ is obtained from $X \sqcup X$ by attaching cells, and is therefore a cofibration.

Actually, the standard sort of homotopy $X \times I \to Y$ for a more general space X is more properly described as a right homotopy.

2) There are lots of cylinder objects, since the map $\nabla : X \sqcup X \to X$ has a factorization as a cofibration followed by a trivial fibration, by **CM5**.

I've used the word "dual", and here is what I mean by that:

Lemma 7.8. *Suppose that $\mathcal{M}$ is a closed model category. Say that a morphism $f^{op} : Y \to X$ of the opposite category $\mathcal{M}^{op}$ is a fibration (resp. cofibration, weak equivalence) if and only if the corresponding map $f : X \to Y$ is a cofibration (resp. fibration, weak equivalence) of $\mathcal{M}$. Then with these definitions, $\mathcal{M}^{op}$ satisfies the axioms for a closed model category.*

Proof. Exercise. □

Thus, if you reverse the arrows in a cylinder object you get a path object, and vice versa. Generally all facts about the homotopical algebra of a model category $\mathcal{M}$ have an equivalent dual assertion in $\mathcal{M}^{op}$.

Examples: In Lemma 7.1, statement 3) is the dual of statement 1), and statement 4) is the dual of statement 2).

Lemma 7.9. *Right homotopy of maps $X \to Y$ is an equivalence relation if Y is fibrant.*

The dual of Lemma 7.9 is the following:

Lemma 7.10. *Left homotopy of maps $X \to Y$ is an equivalence relation if X is cofibrant.*

Proof. Lemma 7.10 is equivalent to Lemma 7.9 in $\mathcal{M}^{op}$. ∎

Proof of Lemma 7.9. First of all, note that that if Y if fibrant then any projection $X \times Y \to X$ is a fibration. It follows that if

$$\begin{array}{ccc} & & Y^I \\ & \overset{s}{\nearrow} & \downarrow{\scriptstyle (p_0,p_1)} \\ Y & \xrightarrow[\Delta]{} & Y \times Y \end{array}$$

is a path object for a fibrant object Y, then the maps p_0 and p_1 are trivial fibrations.

Suppose given right homotopies

$$\begin{array}{ccc} & & Y^I \\ & \overset{h_1}{\nearrow} & \downarrow{\scriptstyle (p_0,p_1)} \\ X & \xrightarrow[(f_1,f_2)]{} & Y \times Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & Y^J \\ & \overset{h_2}{\nearrow} & \downarrow{\scriptstyle (q_0,q_1)} \\ X & \xrightarrow[(f_2,f_3)]{} & Y \times Y \end{array}$$

Form the pullback

$$\begin{array}{ccc} Y^I \times_Y Y^J & \xrightarrow{p_*} & Y^J \\ {\scriptstyle q_*}\downarrow & & \downarrow{\scriptstyle q_0} \\ Y^I & \xrightarrow[p_1]{} & Y \end{array}$$

Then the diagram

$$\begin{array}{ccc} Y^I \times_Y Y^J & \xrightarrow{p_*} & Y^J \\ {\scriptstyle (q_*, q_1 p_*)} \downarrow & & \downarrow {\scriptstyle (q_0, q_1)} \\ Y^I \times Y & \xrightarrow[p_1 \times 1]{} & Y \times Y \end{array}$$

is a pullback and $p_0 \times 1 : Y^I \times Y \to Y \times Y$ is a fibration, so that the composite

$$Y^I \times_Y Y^J \xrightarrow{(p_0 q_*, q_1 p_*)} Y \times Y$$

is a fibration. The weak equivalences s, s' from the respective path objects determine a commutative diagram

$$\begin{array}{ccc} & & Y^I \times_Y Y^J \\ & \nearrow^{(s,s')} & \downarrow {\scriptstyle (p_0 q_*, q_1 p_*)} \\ Y & \xrightarrow[\Delta]{} & Y \times Y \end{array}$$

and the map (s, s') is a weak equivalence since $p_0 q_*$ is a trivial fibration. It follows that the homotopies h, h' determine a right homotopy

$$\begin{array}{ccc} & & Y^I \times_Y Y^J \\ & \nearrow^{(h,h')} & \downarrow {\scriptstyle (p_0 q_*, q_1 p_*)} \\ X & \xrightarrow[(f_1, f_3)]{} & Y \times Y \end{array}$$

We have proved that the right homotopy relation is transitive. It is symmetric since the twist isomorphism $Y \times Y \to Y \times Y$ is a fibration, and it is reflexive since the morphism s in a path object is a right homotopy from the identity to itself. $\square$

Now here's the result that ties the homotopical room together:

Lemma 7.11. *1) Suppose that Y is fibrant and that $X \otimes I$ is a fixed choice of cylinder object for an object X. Suppose that the maps $f, g : X \to Y$ are right homotopic. Then there is a left homotopy*

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{(f,g)} & Y \\ {\scriptstyle i}\downarrow & \nearrow_{h} & \\ X \otimes I & & \end{array}$$

2) Suppose that X is cofibrant and that Y^I is a fixed choice of path object for an object Y. Suppose that the maps $f, g : X \to Y$ are left homotopic. Then there is a right homotopy

$$\begin{array}{ccc} & & Y^I \\ & {}^{h}\nearrow & \downarrow{\scriptstyle p} \\ X & \xrightarrow[(f,g)]{} & Y \times Y \end{array}$$

Proof. Statement 2) is the dual of statement 1). We'll prove statement 1).

Suppose that

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\nabla} & X \\ {\scriptstyle i}\downarrow & \nearrow_{\sigma} & \\ X \otimes I & & \end{array} \quad \text{and} \quad \begin{array}{ccc} & & Y^I \\ & \nearrow^{s} & \downarrow{\scriptstyle (p_0,p_1)} \\ Y & \xrightarrow[\Delta]{} & Y \times Y \end{array}$$

are the fixed choice of cylinder and the path object involved in the right homotopy $f \sim_r g$, respectively, and let $h : X \to Y^I$ be the right homotopy. Form the diagram

$$\begin{array}{ccccc} X \sqcup X & \xrightarrow{(sf,h)} & Y^I & \xrightarrow{p_1} & Y \\ {\scriptstyle i}\downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle p_0} & & \\ X \otimes I & \xrightarrow[f\sigma]{} & Y & & \end{array}$$

The lift θ exists because p_0 is a trivial fibration since Y is fibrant. Then the composite $p_1\theta$ is the desired left homotopy. □

Corollary 7.12. *Suppose that* $f, g : X \to Y$ *are morphisms of* $\mathcal{M}$*, where* X *is cofibrant and* Y *is fibrant. Suppose that*

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\nabla} & X \\ {\scriptstyle i}\downarrow & \nearrow_{\sigma} & \\ X \otimes I & & \end{array} \quad \textit{and} \quad \begin{array}{ccc} & & Y^I \\ & \nearrow^{s} & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[\Delta]{} & Y \times Y \end{array}$$

are fixed choices of cylinder and path objects for X *and* Y *respectively. Then the following are equivalent:*

- *f is left homotopic to g.*
- *There is a right homotopy $h : X \to Y^I$ from f to g.*
- *f is right homotopic to g.*
- *There is a left homotopy $H : X \otimes I \to Y$ from f to g.*

Thus, if X is cofibrant and Y is fibrant, all notions of homotopy of maps $X \to Y$ collapse to the same thing. I shall typically write $f \sim g$ to say that f is homotopic to g (by whatever means) in this case.

Here's the first big application:

Theorem 7.13 (Whitehead Theorem)**.** *Suppose that $f : X \to Y$ is a weak equivalence, and that the objects X and Y are both fibrant and cofibrant. Then f is a homotopy equivalence.*

Proof. It suffices to assume that f is a trivial fibration: every weak equivalence can be factorized as the composite of a trivial fibration with a trivial cofibration, and the trivial cofibration case is dual.

The object Y is cofibrant, so the lifting exists in the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & \overset{j}{\nearrow} & \downarrow f \\ Y & \xrightarrow[1]{} & Y \end{array}$$

Suppose that

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\nabla} & X \\ i \downarrow & \overset{\sigma}{\nearrow} & \\ X \otimes I & & \end{array}$$

is a cylinder object for X, and then form the diagram

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{(jf,1)} & X \\ i \downarrow & \overset{h}{\nearrow} & \downarrow f \\ X \otimes I & \xrightarrow[f\sigma]{} & Y \end{array}$$

The indicated lift (and required homotopy) exists because f is a trivial fibration. $\square$

Examples: 1) Every weak equivalence $f : C \to D$ in $Ch_+(R)$ between complexes of projective R-modules is a chain homotopy equivalence.

2) (traditional Whitehead Theorem) Every weak equivalence $f : X \to Y$ between CW-complexes is a homotopy equivalence.

8 The homotopy category

Here's a construction. For all $X \in \mathcal{M}$ find maps

$$X \xleftarrow{p_X} QX \xrightarrow{j_X} RQX$$

such that

- p_X is a trivial fibration and QX is cofibrant, and j_X is a trivial cofibration and RQX is fibrant (and cofibrant),
- $QX = X$ and $p_X = 1_X$ if X is cofibrant, and $RQX = QX$ and $j_X = 1_{QX}$ if QX is fibrant.

Then any map $f : X \to Y$ determines a diagram

$$\begin{array}{ccccc} X & \xleftarrow{p_X} & QX & \xrightarrow{j_X} & RQX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ Y & \xleftarrow[p_Y]{} & QY & \xrightarrow[j_Y]{} & RQY \end{array}$$

since QX is cofibrant and RQY is fibrant.

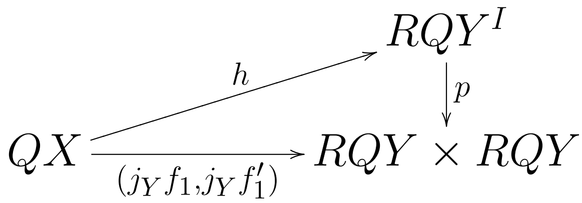
Lemma 8.1. *The map f_2 is uniquely determined up to homotopy.*

Proof. Suppose that f_1' and f_2' are different choices for f_1 and f_2 respectively. Then there is a diagram

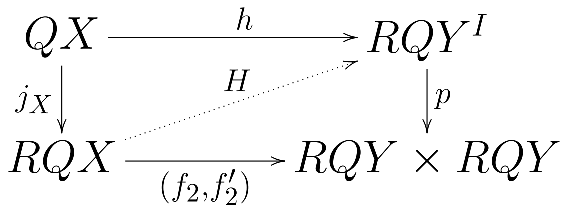
$$\begin{array}{ccccc} QX \sqcup QX & & \xrightarrow{(f_1, f_1')} & & QY \\ {\scriptstyle i}\downarrow & & \nearrow & & \downarrow{\scriptstyle p_Y} \\ QX \otimes I & \xrightarrow[\sigma]{} & QX & \xrightarrow[f p_X]{} & Y \end{array}$$

for any choice of cylinder $QX \otimes I$ for QX, so that f_1 and f_1' are left homotopic.

The maps $j_Y f_1$ and $j_Y f_1'$ are left homotopic, hence right homotopic because QX is cofibrant and RQY is fibrant. Thus, there is a right homotopy



for some (actually any) path object RQY^I. Form the diagram



Then f_2 and f_2' are homotopic. □

Write $\pi(\mathcal{M})_{cf}$ for the category whose objects are the cofibrant-fibrant objects of $\mathcal{M}$, and whose morphisms are homotopy classes of maps. It is a consequence of Lemma 8.1 that there is a well-defined functor

$$\mathcal{M} \to \pi(\mathcal{M})_{cf}$$

defined by $X \mapsto RQX$ and $f \mapsto [RQf]$, where $RQ(f) = f_2$ in the diagram above "defines" $RQ(f)$.

The *homotopy category* $\mathrm{Ho}(\mathcal{M})$ of $\mathcal{M}$ has the same objects as $\mathcal{M}$, and has

$$\hom_{\mathrm{Ho}(\mathcal{M})}(X,Y) = \hom_{\pi(\mathcal{M})_{cf}}(RQX, RQY).$$

There is a functor

$$\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$$

which is the identity on objects, and sends $f : X \to Y$ to the homotopy class $[RQ(f)]$.

NB: The functor γ takes weak equivalences to isomorphisms in $\mathrm{Ho}(\mathcal{M})$, by the Whitehead Theorem (Theorem 7.13).

Lemma 8.2. *Suppose that $f : RQX \to RQY$ represents a morphism $[f] : X \to Y$ of $\mathrm{Ho}(\mathcal{M})$. Then there is a commutative diagram*

$$\begin{array}{ccccc} X & \xleftarrow{\gamma(p_X)} & QX & \xrightarrow{\gamma(j_X)} & RQX \\ {\scriptstyle [f]}\downarrow & & \downarrow{\scriptstyle [f]} & & \downarrow{\scriptstyle \gamma(f)} \\ Y & \xleftarrow[\gamma(p_Y)]{} & QY & \xrightarrow[\gamma(j_Y)]{} & RQY \end{array}$$

in $\mathrm{Ho}(\mathcal{M})$.

Proof. The maps $\gamma(p_X)$ and $\gamma(j_X)$ are isomorphisms defined by the class $[1_{RQX}]$ in $\pi(\mathcal{M})_{cf}$. □

Theorem 8.3. *Suppose that $\mathcal{M}$ is a closed model category, and that a functor $F : \mathcal{M} \to D$ takes weak equivalences to isomorphisms. Then there is a unique functor $F_* : \mathrm{Ho}(\mathcal{M}) \to D$ such that the diagram of functors*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\gamma} & \mathrm{Ho}(\mathcal{M}) \\ & {\scriptstyle F}\searrow & \downarrow {\scriptstyle F_*} \\ & & D \end{array}$$

commutes.

Proof. This result is a corollary of Lemma 8.2. □

Remarks: 1) The category $\mathrm{Ho}(\mathcal{M})$ is a small model for the category $\mathcal{M}[WE]^{-1}$ that is obtained from $\mathcal{M}$ by formally inverting the weak equivalences.

2) The functor $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ induces a fully faithful functor $\gamma_* : \pi(\mathcal{M}_{cf}) \to \mathrm{Ho}(\mathcal{M})$. Further, every object of $\mathrm{Ho}(\mathcal{M})$ is isomorphic to a (cofibrant fibrant) object in the image of γ_*. It follows that the functor γ_* is an equivalence of categories. This observation specializes to well known phenomena:

- The homotopy category of **CGHaus** is equivalent to the category of CW-complexes and ordinary homotopy classes of maps between them.

- The derived category of $Ch_+(R)$ is equivalent to the category of chain complexes of projectives and chain homotopy classes of maps between them.

One final thing: in general, the functor $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ reflects weak equivalences:

Proposition 8.4. *Suppose that $\mathcal{M}$ is a closed model category, and that $f : X \to Y$ is a morphism such that $\gamma(f)$ is an isomorphism in $\mathrm{Ho}(\mathcal{M})$. Then f is a weak equivalence of $\mathcal{M}$.*

For the proof, it is enough to suppose that both X and Y are fibrant and cofibrant and that f is a fibration with a homotopy inverse $g : Y \to X$. Then the idea is to show that f is a weak equivalence.

This claim is a triviality in almost all cases of interest, but it is a bit tricky to prove in full generality. This result appears in [1] as Proposition II.1.14.

References

[1] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.