Lecture 004 (October 19, 2009)

9 Torsion products: basic theory

Suppose that M is a right R-module and that N is a left R-module. The *tensor product* $M \otimes_R N$ is the abelian group freely generated by the pairs $(m, n) \in M \times N$, subject to the relations

- $(m + m', n) \sim (m, n) + (m', n)$
- $(m, n + n') \sim (m, n) + (m, n')$
- $(mr, n) = (m, rn)$ for $r \in R$.

The class of a generator (m, n) in $M \otimes_R N$ is written $m \otimes n$. The elements $m \otimes n$ generate the group $M \otimes_R N$.

Suppose that $f : M \times N \to K$ is a function taking values in an abelian group K which satisfies the following identities:

- $f(m + m', n) \sim f(m, n) + f(m', n)$
- $f(m, n + n') \sim f(m, n) + f(m, n')$
- $f(mr, n) = f(m, rn)$ for $r \in R$.

Such a function is said to be *R-bilinear*.

Every R-bilinear function f extends to a unique abelian group homomorphism $f_* : M \otimes_R N \to K$ such that

$$f_*(m \otimes n) = f(m, n).$$

It follows that the R-bilinear map

$$M \times N \to M \otimes_R N$$

defined by $(m, n) \mapsto m \otimes n$ is the initial R-bilinear map which is defined on $M \times N$.

Every right R-module map $f : M \to M'$ and every left R-module map $g : N \to N'$ together induce a homomorphism

$$f \otimes g : M \otimes_R N \to M' \otimes_R N'$$

which is defined by

$$(f \otimes g)(m \otimes n) = f(m) \otimes g(n).$$

The assigment $(M, N) \mapsto M \otimes_R N$ uniquely specifies $M \otimes_R N$ as an abelian group, and the tensor product construction is functorial in M and N.

Lemma 9.1. *1) The module structure maps $M \times R \to M$ and $R \times N \to N$ induce canonical isomorphisms*

$$M \otimes_R R \xrightarrow{\cong} M, \quad R \otimes_R N \xrightarrow{\cong} N.$$

2) Suppose that M' and N' are second choices of left and right R-modules. Then there are canonical isomorphisms

$$(M \oplus M') \otimes_R N \xrightarrow{\cong} (M \otimes_R N) \oplus (M' \otimes_R N),$$

$$M \otimes_R (N \oplus N') \xrightarrow{\cong} (M \otimes_R N) \oplus (M \otimes_R N')$$

which are induced by projections.

The proof of this Lemma is an exercise.

Lemma 9.2. *1) Suppose that*

$$M \xrightarrow{f} M' \xrightarrow{\pi} M'' \to 0$$

is an exact sequence of right R-modules. Then the induced sequence of abelian group homomorphisms

$$M \otimes_R N \xrightarrow{f\otimes 1} M' \otimes_R N \xrightarrow{p\otimes 1} M'' \otimes_R N \to 0$$

is exact.

2) Suppose that

$$N \to N' \to N'' \to 0$$

is an exact sequence of left R-modules. Then the induced sequence

$$M \otimes_R N \to M \otimes_R N' \to M \otimes_R N'' \to 0$$

is exact.

Proof. We'll prove statement 1). The proof of statement 2) is similar.

Form the exact sequence

$$M \otimes_R N \xrightarrow{f\otimes 1} M' \otimes_R N \xrightarrow{\pi} C \to 0,$$

so that π is the cokernel of $f \otimes 1$. Then

$$(p \otimes 1) \cdot (f \otimes 1) = 0$$

so there is a unique homomorphism p_* which makes the diagram

$$\begin{array}{ccc} M' \otimes_R N & \xrightarrow{\pi} & C \\ & \searrow^{p\otimes 1} & \downarrow p_* \\ & & M'' \otimes_R N \end{array}$$

commute. The function

$$M'' \times N \xrightarrow{\sigma} C$$

which is given by $([m], n) \mapsto \pi(m, n)$ (here $[m] = p(m)$) is well defined and and R-bilinear, and induces a unique abelian group homomorphism $\sigma_* : M'' \otimes_R N \to C$. Use universal properties to show that σ_* is the inverse abelian group homomorphism of p_*. □

Lemma 9.2 says that the tensor product functor is *right exact*, in both variables. This functor does not preserve monomorphisms in general, so it fails to be left exact, and is therefore not exact: it does not preserve exact sequences.

Example 9.3. Suppose that $n > 1$ is an integer. Then multiplication by n defines an abelian group homomorphism

$$\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}$$

which is injective. According to Lemma 9.1 there is a commutative diagram

$$\begin{array}{ccc} \mathbb{Z} \otimes \mathbb{Z}/n & \xrightarrow{(\times n)\otimes 1} & \mathbb{Z} \otimes \mathbb{Z}/n \\ \cong \downarrow & & \downarrow \cong \\ \mathbb{Z}/n & \xrightarrow[\times n]{} & \mathbb{Z}/n \end{array}$$

and the map $\times n : \mathbb{Z}/n \to \mathbb{Z}/n$ is the 0-map, which is certainly not injective.

Example 9.4. Lemma 9.1 implies that the inclusion of a direct summand

$$M \to M \oplus M'$$

of right R-modules induces a monomorphism

$$M \otimes_R N \to (M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N),$$

since the displayed composite is the inclusion of a direct summand.

Similarly, if

$$N \to N \oplus N'$$

is an inclusion of a direct summant of left R-modules, then the induced map

$$M \otimes_R N \to M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')$$

is a monomorphism.

We also have the following:

Lemma 9.5. *1) If P is a projective right R-module, then the functor defined on left R-modules by $N \mapsto P \otimes_R N$ is exact.*

2) If Q is a projective left R-module, then the functor defined on right R-modules by $M \mapsto M \otimes_R Q$ is exact.

Proof. Again, we'll just prove statement 1), since statement 2) is similar.

The functor

$$N \mapsto R \otimes_R N \cong N$$

is exact by Lemma 9.1. If F is a finitely generated free module, then

$$F \cong \oplus_{i=1}^n R$$

is a finite direct sum of copies of R, and finite direct sums of exact functors are exact (exercise), so that tensoring with F is exact. Every free R-module F is a filtered colimit of finitely generated free modules (exercise) and filtered colimits of exact functors are exact (exercise), so that tensoring with an arbitrary free module F is exact. Every projective right R-module P is a retract (aka. direct summand) of a free module, and retracts of exact functors are exact (exercise). $\square$

The failure of left exactness for the functor $M \mapsto M \otimes_R N$ is encoded in the higher torsion products $\mathrm{Tor}_n(M, N)$, $n \geq 0$, which are the left derived functors of the tensor product.

We can define these functors quite generally as follows. Suppose that C is an ordinary chain complex of right R-modules and that N is a left R-module, as above.

There is a functorial cofibrant model $\pi_C : P_C \to C$ (π_C is a weak equivalence, actually a trivial fibration, with P_C cofibrant) for the chain complex C. The functoriality of the cofibrant model comes from the small object argument for the model structure on the chain complex category $Ch_+(R)$, which argument is natural in chain complex maps. Recall that the cofibrant model P_C is a chain complex of projective right R-modules. We define the *higher torsion product* functors $\mathrm{Tor}_n(C,N)$ by

$$\mathrm{Tor}_n(C,N) = H_n(P_C \otimes_R N),\ n \geq 0.$$

Any chain complex morphism $f : C \to D$ induces a commutative diagram

$$\begin{array}{ccc} P_C & \xrightarrow{f_*} & P_D \\ {\scriptstyle \pi_C}\downarrow & & \downarrow{\scriptstyle \pi_D} \\ C & \xrightarrow[f]{} & D \end{array}$$

and there is an induced chain complex map

$$f_* \otimes 1 : P_C \otimes_R N \to P_D \otimes N,$$

with corresponding induced morphisms

$$f_* : \mathrm{Tor}_n(C,N) \to \mathrm{Tor}_n(D,N)$$

for all $n \geq 0$. These last maps define the functors $\mathrm{Tor}_*(\ ,N)$. Observe (exercise) that a morphism $g : N \to N'$ also induces chain maps

$$P_C \otimes_R N \xrightarrow{1 \otimes g} P_C \otimes_R N'$$

and hence abelian group homomorphisms

$$g_* : \mathrm{Tor}_n(C,N) \to \mathrm{Tor}_n(C,N'),\ n \geq 0.$$

The assignment

$$(C,N) \mapsto \mathrm{Tor}_n(C,N)$$

is therefore functorial in both chain complexes C and modules N. This pairing is actually a bifunctor in the sense that the functors in C and N commute with each other.

Remark 9.6. Here is a fundamental observation: we have used a functorial cofibrant model for the chain complex C, but we don't need to if we are only interested in specifying the functors $\mathrm{Tor}_*(C, N)$ up to isomorphism.

Suppose that $f : Q \to C$ is a weak equivalence with Q cofibrant, and find a factorization

$$\begin{array}{ccc} Q & \xrightarrow{i} & Q' \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & C \end{array}$$

such that i is a trivial cofibration and p is a trivial fibration. Then Q' is cofibrant. Also, the lifting exists in the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & Q' \\ \downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle p} \\ P_C & \xrightarrow[\pi_C]{} & C \end{array}$$

since p is a trivial fibration and P_C is cofibrant. Then also θ is a weak equivalence since π_C and p

are weak equivalences. The maps

$$P_C \xrightarrow{\theta} P \xleftarrow{i} Q$$

are weak equivalences of cofibrant (and fibrant) chain complexes, and are therefore chain homotopy equivalences. It follows that the induced maps

$$P_C \otimes_R N \xrightarrow{\theta\otimes 1} P \otimes_R N \xleftarrow{i\otimes 1} Q \otimes_R N$$

are chain homotopy equivalences. These maps are therefore homology isomorphisms (by Corollary 9.8 below), and so they induce isomorphisms

$$\mathrm{Tor}_n(C, N) = H_n(P_C \otimes_R N) \xrightarrow{\cong} H_n(P \otimes_n N) \xleftarrow{\cong} H_n(Q \otimes_R N).$$

Lemma 9.7. *Suppose that the chain maps $f, g : C \to D$ are chain homotopic. Then the induced maps $f_*, g_* : H_n(C) \to H_n(D)$ are equal.*

Proof. Recall that a chain homotopy from f to g consists of homomorphisms $s : C_n \to D_{n+1}$ such that

$$f - g = \partial s + s\partial : C_n \to D_n,\ n \geq 0.$$

Thus, if z is an n-cycle, then

$$f(z) - g(z) = \partial s(z) + s\partial(z) = \partial s(z)$$

is a boundary, so that

$$[f(z)] = [g(z)] \in H_n(D).$$

□

Corollary 9.8. *Suppose that $f : C \to D$ is a chain homotopy equivalence. Then f induces isomorphisms*

$$H_n(C) \xrightarrow{\cong} H_n(D)$$

for all $n \geq 0$.

Proof. When we say that f is a chain homotopy equivalence, we mean that there is a chain map $g : D \to C$ such that fg is chain homotopic to 1_D and gf is chain homotopic to 1_C. It follows from Lemma 9.7 that the induced maps f_* and g_* in homology are inverse to each other. □

The functors $\mathrm{Tor}_n(C, N)$, as defined here, are actually *hypertor* functors from a classical point of view, but they specialize to the standard higher derived functors of the tensor product.

In effect, suppose that M is a right R-module, and identify it with the chain complex $M[0]$ which consists of M concentrated in degree 0. A cofibrant model $P \to M[0]$ is a projective resolution of M in the classical sense, and the higher torsion products $\mathrm{Tor}_n(M[0], N)$ can be identified up to isomorphism with the homology groups

$$H_n(P \otimes_R M),$$

on account of the observations in Remark 9.6. One usually writes

$$\mathrm{Tor}_n(M, N) := \mathrm{Tor}_n(M[0], N).$$

Lemma 9.9. *1) There is a natural isomorphism*

$$\mathrm{Tor}_0(C, N) \cong H_0(C) \otimes_R N.$$

2) There is a natural isomorphism

$$\mathrm{Tor}_0(M, N) \cong M \otimes_R N.$$

Proof. The second statement follows from the first (exercise).

Suppose that $\pi : P \to C$ is a cofibrant model of the complex C. Then the sequence

$$P_1 \xrightarrow{\partial} P_0 \xrightarrow{\pi} H_0(C) \to 0$$

is exact, so the sequence

$$P_1 \otimes_R N \xrightarrow{\partial \otimes 1} P_0 \otimes_R N \xrightarrow{\pi \otimes 1} H_0(C) \otimes_R N \to 0$$

is exact, by Lemma 9.2. It follows that there are isomorphisms

$$\mathrm{Tor}_0(C, N) \cong H_0(P \otimes_R M) \cong H_0(C) \otimes_R N.$$

□

Lemma 9.10. *Suppose that*

$$0 \to F \xrightarrow{j} E \xrightarrow{p} B \to 0$$

is a short exact sequence of chain complexes. Then there is an induced long exact sequence

$$\dots \xrightarrow{\partial} \mathrm{Tor}_n(F,N) \xrightarrow{i_*} \mathrm{Tor}_n(E,N) \xrightarrow{p_*} \mathrm{Tor}_n(B,N) \xrightarrow{\partial}$$
$$\dots \xrightarrow{\partial} \mathrm{Tor}_0(F,N) \xrightarrow{i_*} \mathrm{Tor}_0(E,N) \xrightarrow{p_*} \mathrm{Tor}_0(B,N) \to 0.$$

This long exact sequence is functorial in short exact sequences of chain complexes.

Proof. Choose a functorial cofibrant model $\pi_B : P_B \to B$, and form the pullback diagram

$$\begin{array}{ccc} E' & \xrightarrow{p_*} & P_B \\ {\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle \pi_B} \\ E & \xrightarrow[p]{} & B \end{array}$$

Then p_* is surjective in all degrees and π_* is a trivial fibration. Choose a functorial cofibrant model $\pi_{E'} : P_{E'} \to E'$. Then the composite $p_*\pi_{E'} : P_{E'} \to P_B$ is surjective in all degrees, and the composite $\pi_*\pi_{E'} : P_{E'} \to E$ is a trivial fibration. Suppose that $j' : F' \to P_{E'}$ is the inclusion of the kernel of the map $p_*\pi_{E'}$. Then in the induced comparison of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & F' & \xrightarrow{j'} & P_{E'} & \xrightarrow{p_*\pi_{E'}} & P_B & \to & 0 \\ & & {\scriptstyle \pi'}\downarrow & & \simeq\downarrow{\scriptstyle \pi_*\pi_{E'}} & & \simeq\downarrow{\scriptstyle \pi_B} & & \\ 0 & \to & F & \xrightarrow[j]{} & E & \xrightarrow[p]{} & B & \to & 0 \end{array}$$

the induced map π' is a weak equivalence. Furthermore, the epimorphism $p_*\pi_{E'} : P_{E'} \to P_B$ is degreewise split since P_B is a complex of projective modules. It follows that F is a complex of projective modules, and is therefore cofibrant. It also follows that the sequence of chain complex morphisms

$$0 \to F' \otimes_R N \to P_{E'} \otimes_R N \to P_B \otimes N \to 0 \quad (1)$$

is degreewise split, and is therefore exact. The desired long exact sequence is the long exact sequence in homology which is associated to the short exact sequence (1). □

Corollary 9.11. *Suppose that*

$$0 \to M \xrightarrow{j} M' \xrightarrow{p} M'' \to 0$$

is a short exact sequence of right R-modules, and that N is a left R-module. Then there is an induced functorial long exact sequence

$$\dots \xrightarrow{\partial} \mathrm{Tor}_n(M,N) \xrightarrow{i_*} \mathrm{Tor}_n(M',N) \xrightarrow{p_*} \mathrm{Tor}_n(M'',N) \xrightarrow{\partial}$$
$$\dots \xrightarrow{\partial} M \otimes_R N \xrightarrow{i_*} M' \otimes_R N \xrightarrow{p_*} M'' \otimes_R N \to 0.$$

Lemma 9.12. *Suppose that C is a chain complex of right R-modules and that*

$$0 \to N \xrightarrow{i} N' \xrightarrow{p} N'' \to 0$$

is a short exact sequence of left R-modules. Then there is an induced functorial long exact sequence

$$\dots \xrightarrow{p_*} \mathrm{Tor}_{n+1}(C, N'') \xrightarrow{\partial} \mathrm{Tor}_n(C, N) \xrightarrow{i_*} \mathrm{Tor}_n(C, N') \xrightarrow{p_*} \dots$$
$$\dots \xrightarrow{\partial} \mathrm{Tor}_0(C, N) \xrightarrow{i_*} \mathrm{Tor}_0(C, N') \xrightarrow{p_*} \mathrm{Tor}_0(C, N'') \to 0.$$

The proof of this result is an exercise. Use Lemma 9.5.

Corollary 9.13. *Suppose that M is a right R-module and that*

$$0 \to N \xrightarrow{i} N' \xrightarrow{p} N'' \to 0$$

is a short exact sequence of left R-modules. Then there is an induced functorial long exact sequence

$$\dots \xrightarrow{p_*} \mathrm{Tor}_{n+1}(M, N'') \xrightarrow{\partial} \mathrm{Tor}_n(M, N) \xrightarrow{i_*} \mathrm{Tor}_n(M, N') \xrightarrow{p_*} \dots$$
$$\dots \xrightarrow{\partial} M \otimes_R N \xrightarrow{i_*} M \otimes_R N' \xrightarrow{p_*} M \otimes_R N'' \to 0.$$

Examples:

1) Suppose that P is a projective right R-module, and that N is an arbitrary left R-module. Then the complex

$$\dots \to 0 \to 0 \to P$$

is a projective resolution of P, so that the groups $\mathrm{Tor}_n(P, N)$ are the homology groups of the complex

$$\cdots \to 0 \to 0 \to P \otimes_R N.$$

It follows that

$$\mathrm{Tor}_n(P, N) \cong \begin{cases} 0 & \text{if } n > 0, \text{ and} \\ P \otimes_R N & \text{if } n = 0. \end{cases}$$

Suppose that $P \to M[0]$ is a projective resolution of M in right R-modules, and that Q is a projective left R-module. Then the sequence

$$\cdots \to P_1 \otimes_R Q \to P_0 \otimes_R Q \to M \otimes_R Q \to 0$$

is exact, so that

$$\mathrm{Tor}_n(M, Q) \cong \begin{cases} 0 & \text{if } n > 0, \text{ and} \\ M \otimes_R Q & \text{if } n = 0. \end{cases}$$

2) The short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n \to 0$$

defines a projective resolution

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z} \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/n \end{array}$$

of the abelian group $\mathbb{Z}/n$ (aka. $\mathbb{Z}$-module) in the category of abelian groups. Suppose that A is an arbitrary abelian group. Then the tensor product of the resolution above with A is isomorphic to the chain complex

$$\cdots \to 0 \to 0 \to A \xrightarrow{\times n} A.$$

It follows that

$$\begin{aligned}
&\mathrm{Tor}_0(\mathbb{Z}/n, A) \cong \mathbb{Z}/n \otimes A \cong A/nA, \\
&\mathrm{Tor}_1(\mathbb{Z}/n, A) \cong \ker(A \xrightarrow{\times n} A), \\
&\mathrm{Tor}_i(\mathbb{Z}/n, A) = 0, \ \ i > 1.
\end{aligned}$$

Here,

$$\mathrm{Tor}_1(\mathbf{Z}/n, A) = \mathrm{Tor}(\mathbb{Z}/n, A) = {}_nA$$

are standard notations for the same thing, which is the subgroup of n-torsion elements of A.

3) Suppose that B is an abelian group, and let $p : F \to B$ be a surjective homomorphism, where F is a free abelian group. The kernel K of p is a subgroup of a free abelian group, and is therefore free (since $\mathbb{Z}$ is a principal ideal domain). It follows that the short exact sequence

$$0 \to K \to F \to B \to 0$$

defines a projective resolution of B in the category of abelian groups. Thus the groups $\mathrm{Tor}_n(B, A)$ are the homology groups of the complex

$$\cdots \to 0 \to K \otimes A \to F \otimes A,$$

and so $\operatorname{Tor}_n(B, A) = 0$ for $n > 1$, for all abelian groups B and A.

4) Suppose that $\mathbb{Z}(\mathbb{Z}/m)$ is the group algebra for the cyclic group $\mathbb{Z}/m$ over the integers. Suppose that t is the multiplicative generator of the group $\mathbb{Z}/m$, so that all elements of $\mathbb{Z}(\mathbb{Z}/m)$ have the form

$$a_0 + a_1 t + a_2 t^2 + \cdots + a_{m-1} t^{m-1},$$

with $a_i \in \mathbb{Z}$. Write

$$N = 1 + t + t^2 + \cdots + t^{m-1}$$

for the so-called *norm element*. Then

$$(1-t)N = (1-t)(1 + t + \cdots + t^{m-1}) = 0$$

in $\mathbb{Z}(\mathbb{Z}/m)$ (exercise).

The sequence of $\mathbb{Z}(\mathbb{Z}/m)$-modules

$$\mathbb{Z}(\mathbb{Z}/m) \xrightarrow{N} \mathbb{Z}(\mathbb{Z}/m) \xrightarrow{(1-t)} \mathbb{Z}(\mathbb{Z}/m) \xrightarrow{N} \mathbb{Z}(\mathbb{Z}/m)$$

is exact. To see this, observe that

$$N(a_0 + a_1 t + \cdots + a_{m-1} t^{m-1}) = (a_0 + \cdots + a_{m-1})N$$

and that

$$(1-t)(b_0 + b_1 t + \cdots + b_{m-1} t^{m-1}) = 0$$

if and only $b_0 = b_1 = \cdots = b_{m-1}$. Then

- $N(a_0 + \cdots + a_{m-1}t^{m-1}) = 0$ if and only if $a_0 + \cdots + a_{m-1} = 0$, and in that case

$$(1-t)(a_0 + (a_0 + a_1)t + \cdots + (a_0 + \cdots + a_{m-1})t^{m-1})$$
$$= a_0 + a_1 t + \cdots + a_{m-1}t^{m-1}.$$

- If $b_0 = b_1 = \dots b_{m-1} = b$ then

$$N \cdot b = b_0 + b_1 t + \cdots + b_{m-1}t^{m-1}.$$

It follows that the sequence

$$\dots \xrightarrow{(1-t)} \mathbb{Z}(\mathbb{Z}/m) \xrightarrow{N} \mathbb{Z}(\mathbb{Z}/m) \xrightarrow{(1-t)} \mathbb{Z}(\mathbb{Z}/m) \xrightarrow{\epsilon} \mathbb{Z} \to 0 \quad (2)$$

is a projective resolution of the trivial module $\mathbb{Z}$ in the category of $\mathbb{Z}(\mathbb{Z}/m)$-modules. Here, ϵ is the map defined by $\epsilon(t^i) = 1$.

Generally, a module M over the group ring $\mathbb{Z}(G)$ is said to be *trivial* if $g \cdot m = m$ for all $g \in G$ and $m \in M$.

Tensor the resolution (2) with the trivial $\mathbb{Z}(\mathbb{Z}/m)$-module $\mathbb{Z}$. The result is isomorphic to the chain complex

$$\dots \xrightarrow{(1-t)} \mathbb{Z} \xrightarrow{N} \mathbb{Z} \xrightarrow{(1-t)} \mathbb{Z},$$

which is the chain complex

$$\dots \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

It follows that

$$\begin{aligned}
&\mathrm{Tor}_0(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z} \otimes_{\mathbb{Z}(\mathbb{Z}/m)} \mathbb{Z} \cong \mathbb{Z} \\
&\mathrm{Tor}_{2i+1}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}/m,\ i \geq 0, \\
&\mathrm{Tor}_{2i}(\mathbb{Z},\mathbb{Z}) = 0,\ i > 0,
\end{aligned}$$

in the category of $\mathbb{Z}(\mathbb{Z}/m)$-modules. One often sees the notations

$$H_i(B(\mathbb{Z}/m),\mathbb{Z}) = H_i(\mathbb{Z}/m,\mathbb{Z}) = \mathrm{Tor}_i(\mathbb{Z},\mathbb{Z})$$

for these higher torsion products.

In general, the i^{th} *homology group* of the group G with coefficients in the G-module A is defined by

$$H_i(G,A) = \mathrm{Tor}_i(\mathbb{Z},A),$$

where $\mathbb{Z}$ has the trivial G-module structure.

Lemma 9.14. *Suppose that $f : C \to D$ is a weak equivalence of chain complexes of right R-modules, and that Q is a projective left R-module. Then the induced chain complex morphism*

$$f \otimes 1 : C \otimes_R Q \to D \otimes_R Q$$

is a weak equivalence.

Proof. Suppose that $f : C \to D$ is a trivial fibration. Then f is an epimorphism in all degrees, with acyclic kernel K. The sequence

$$0 \to K \otimes_R Q \to C \otimes_R Q \to D \otimes_R Q \to 0$$

is exact since Q is projective (Lemma 9.5), and also the chain complex $K \otimes_R Q$ is acyclic since Q is projective (exercise). It follows from Lemma 9.10 that $f \otimes 1$ is a weak equivalence if f is a trivial fibration.

If f is a trivial cofibration, then f is a monomorphism (Corollary 4.9) with acyclic cokernel E. The sequence

$$0 \to C \otimes_R Q \to D \otimes_R Q \to E \otimes_R Q \to 0$$

is exact since Q is projective, and the complex $E \otimes_R Q$ is acyclic. Thus (Lemma 9.10) the map $f \otimes 1$ is a weak equivalence if f is a trivial cofibration.

The general case follows from a factorization argument (ie. use **CM5**). □

We have been resolving the right R-module M to get a definition of the higher torsion products:

$$\mathrm{Tor}_n(M, N) = H_n(P \otimes_R N)$$

where $P \to M$ is a projective resolution of M. The following result says that the same invariant can be computed by resolving the module N instead.

Lemma 9.15. *Suppose that $Q \to N$ is a projective resolution of the left R-module N, and that M is a right R-module. Then there is an isomorphism*

$$\mathrm{Tor}_i(M,N) \cong H_i(M \otimes_R Q)$$

for all $i \geq 0$.

Proof. Suppose that

$$0 \to M' \to P \to M \to 0$$

is an exact sequence of right R-modules with P projective. Then

$$\mathrm{Tor}_i(P,N) = 0 = H_i(P \otimes Q)$$

for $i \geq 1$, and there are natural isomorphisms

$$\mathrm{Tor}_0(M,N) \xrightarrow{\cong} M \otimes_R N \xleftarrow{\cong} H_0(M \otimes_R Q).$$

By comparing the bottom ends of the corresponding long exact sequences, one sees that there is an induced (dotted arrow) isomorphism θ_1 which makes the following diagram commute

$$\begin{array}{ccccccc}
0 \to & \mathrm{Tor}_1(M,N) & \xrightarrow{\partial} & \mathrm{Tor}_0(M',N) & \to & \mathrm{Tor}_0(P,N) \\
& \theta_1 \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
0 \to & H_1(M \otimes_R Q) & \xrightarrow[\partial]{} & H_0(M' \otimes_R Q) & \to & H_0(P \otimes_R Q)
\end{array}$$

In degrees $n \geq 2$, isomorphisms θ_n are constructed inductively so that the diagrams

$$\begin{array}{ccc} \mathrm{Tor}_n(M,N) & \xrightarrow[\cong]{\partial} & \mathrm{Tor}_{n-1}(M',N) \\ {\scriptstyle \theta_n}\downarrow & & \cong \downarrow {\scriptstyle \theta_{n-1}} \\ H_n(M \otimes_R Q) & \xrightarrow[\partial]{\cong} & H_{n-1}(M' \otimes_R Q) \end{array}$$

commute. □

Remark 9.16. The proof of Lemma 9.15 is a bit ad hoc, although it involves a standard technique from homological algebra, which is essentially the foundation of Grothendieck's theory of δ-functors — see [1]. A second proof of this result will appear in the next section.

What follows is one of the fundamental applications of the theory of higher torsion products. There are many others.

Theorem 9.17 (universal coefficients)**.** *Suppose that X is a simplicial set and that A is an abelian group. Then there is a short exact sequence*

$$0 \to H_n(X,\mathbb{Z}) \otimes A \to H_n(X,A) \to \mathrm{Tor}(H_{n-1}(X,\mathbb{Z}),A) \to 0.$$

This sequence is natural in simplicial sets X and abelian groups A.

Proof. Recall that $H_n(X, A)$ is the n^{th} homology group of the complex $\mathbb{Z}(X) \otimes A$, where $\mathbb{Z}(X)$ is the Moore complex of X, which complex consists of free abelian groups in all degrees.

Suppose that the short exact sequence

$$0 \to K \xrightarrow{i} F \xrightarrow{p} A \to A$$

is a free resolution of A as above. Then the induced sequence of chain complex morphisms

$$0 \to \mathbb{Z}(X) \otimes K \xrightarrow{1 \otimes i} \mathbb{Z}(X) \otimes F \xrightarrow{1 \otimes p} \mathbb{Z}(X) \otimes A \to 0 \tag{3}$$

is short exact, since all groups $\mathbf{Z}(X)_n$ are free abelian (Lemma 9.5). There is a commutative diagram

$$\begin{array}{ccc} H_n(\mathbb{Z}(X) \otimes K) & \xrightarrow{(1 \otimes i)_*} & H_n(\mathbb{Z}(X) \otimes F) \\ \cong \uparrow & & \uparrow \cong \\ H_n(X, \mathbb{Z}) \otimes K & \xrightarrow[1 \otimes i]{} & H_n(X, \mathbb{Z}) \otimes F \end{array}$$

since homology commutes with direct sums. The long exact sequence associated to the short exact sequence (3) induces short exact sequences

$$0 \to \text{cok}(1 \otimes i)_* \to H_n(X, A) \to \ker(1 \otimes i)_* \to 0,$$

and there are isomorphisms

$$\operatorname{cok}(1 \otimes i)_* \cong H_n(X, \mathbb{Z}) \otimes A,$$
$$\ker(1 \otimes i)_* \cong \operatorname{Tor}(H_{n-1}(X, \mathbb{Z}), A)$$

for the groups appearing in this sequence. □

References

[1] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.